First-Order Logic

Adapted from Russel and Norvig

Outline

- Why FOL?
- Syntax and semantics of FOL
- Using FOL
- Wumpus world in FOL
- Knowledge engineering in FOL

Pros and cons of propositional logic

☺ Propositional logic is declarative

☺ Propositional logic allows partial/disjunctive/negated information
  - (unlike most data structures and databases)

☺ Propositional logic is compositional:
  - meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

☺ Meaning in propositional logic is context-independent
  - (unlike natural language, where meaning depends on context)

☹ Propositional logic has very limited expressive power
  - (unlike natural language)
  - E.g., cannot say "pits cause breezes in adjacent squares"

Example

- Consider

$$\frac{\text{Katy is a cat} \quad \text{cats are mammals}}{\text{Katy is a mammal}}$$

- In propositional logic this would be represented as

$$\frac{c, m}{k}$$

- This derivation is not valid in propositional logic. If it were then from any c and m could derive any k. We need to capture the connection between c and m.
- We will use *first-order* or *predicate logic*.

First-order logic

- Whereas propositional logic assumes the world contains facts,
- first-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  - Functions: father of, best friend, one more than, plus, …

Syntax of FOL: Basic elements

- Constants KingJohn, 2, NUS,...
- Predicates Brother, >,...
- Functions Sqrt, LeftLegOf,...
- Variables x, y, a, b,...
- Connectives $\neg$, $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
- Equality =
- Quantifiers $\forall$, $\exists$

Example

- lecturer(Schmidt-Thieme,KI)
- male(Schmidt-Thieme)
- < (3, 4)
- < (4, plustwo(1))
- mammal(Katy)
- Schmidt-Thieme, Katy, KI, 3, 4 and 1 are *constants*.
- lecturer, male, mammal, and < are *predicates*.
- male, mammal have *arity* one and the other predicates have arity two.
- plustwo is a function (that refer to other objects).
- For example plustwo(1) refers to the constant 3

Atomic sentences

- *Term* is a logical expression that refers to an object. *Constants,variables* and *functions* are all terms.
- Thus plustwo(1), Schmidt-Thieme and 3 are all terms.
- *Atomic sentences* are predicates applied to a list of terms (in brackets).
- E.g.,
  - male(father_of(Leandro)),
  - cat(Katy),
  - shares_office(Leandro, Christine)
  - and < (3, 4) are all atomic sentences

Complex sentences

- Complex sentences are made from atomic sentences using connectives

  $\neg S$, $S_1 \wedge S_2$, $S_1 \vee S_2$, $S_1 \Rightarrow S_2$, $S_1 \Leftrightarrow S_2$,

E.g. *Sibling(KingJohn,Richard)* $\Rightarrow$ *Sibling(Richard,KingJohn)*

$>(1,2) \vee \leq (1,2)$

$>(1,2) \wedge \neg >(1,2)$

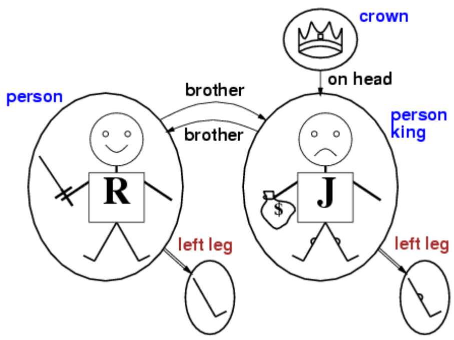
Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains objects (domain elements) and relations among them
- Interpretation specifies referents for
  - constant symbols → objects
  - predicate symbols → relations
  - function symbols → functional relations
- An atomic sentence *predicate*($term_1$,...,$term_n$) is true iff the objects referred to by $term_1$,...,$term_n$ are in the relation referred to by *predicate*

Interpretation

- We need a domain to which we are referring. lecturer(Schmidt-Thieme,KI)
- The name Schmidt-Thieme is mapped to the object in the domain we are referring to (Prof. Lars Schmidt-Thieme).
- The name KI is mapped to the object in the domain we are referring to (the course KI).
- The predicate name lecturer will be mapped to a set of pairs of objects where the first in the pair is the (real) person who teaches the second in the pair.
- Hence the above evaluates to true.

Models for FOL: Example



Quantifiers

- Quantifiers allow us to express properties about collections of objects.
- The quantifiers are
- ∀ universal quantifier *'For all . . . '*
- ∃ existential quantifier *'There exists . . . '*
- If P(x) is a predicate then we can write
- $\forall x\ P(x)$; and
- $\exists x\ P(x)$;
- where x is a *variable* which can stand for any object in
- the domain.

Universal quantification

- ∀<*variables*> <*sentence*>

All kings are persons
$\forall x\ King(x) \Rightarrow Person(x)$

For all *x*, if *x* is a king, then *x* is a person

- $\forall x\ P$ is true in a model *m* iff *P* is true with *x* being each possible object in the model
- Roughly speaking, equivalent to the conjunction of instantiations of *P*

  King(John) ⇒ Person(John)
  ∧ King(Richard) ⇒ Person(Richard)
  ∧ King(Peter) ⇒ Person(Peter)
  ∧ ...

A common mistake to avoid

- Typically, ⇒ is the main connective with ∀
- Common mistake: using ∧ as the main connective with ∀:

  $\forall x\ King(x) \wedge Person(x)$
  means "Everyone is a king and everyone is smart"

Existential quantification

- ∃<*variables*> <*sentence*>

- There exists an x such that x is a man and x is a father
- (some men are fathers)
- $\exists x\ man(x) \wedge father(x)$

- $\exists x\ P$ is true in a model *m* iff *P* is true with *x* being some possible object in the model
- Roughly speaking, equivalent to the disjunction of instantiations of *P*

  man(Peter) ∧ father(Peter)
  ∨ man(John) ∧ father(John)
  ∨ man(Tobias) ∧ father(Tobias)
  ∨ ...

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

  $\exists x\ \ man(x) \Rightarrow father(x)$

  is true if there is anyone who is not a man!

Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$
- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$
- $\exists x\ \forall y$ Loves(x,y)
  - "There is a person who loves everyone in the world"
- $\forall y\ \exists x$ Loves(x,y)
  - "Everyone in the world is loved by at least one person"
- Quantifier duality: each can be expressed using the other
- $\forall x$ Likes(x,IceCream) $\qquad \neg\exists x\ \neg$Likes(x,IceCream)
- $\exists x$ Likes(x,Broccoli) $\qquad \neg\forall x\ \neg$Likes(x,Broccoli)

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
- E.g., definition of *Sibling* in terms of *Parent*:

  $\forall x,y\ Sibling(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f\ \neg (m = f) \wedge$ Parent(m,x) $\wedge$ Parent(f,x) $\wedge$ Parent(m,y) $\wedge$ Parent(f,y)$]$

Using FOL

The kinship domain:

- Brothers are siblings

  $\forall x,y\ Brother(x,y) \Leftrightarrow Sibling(x,y)$
- One's mother is one's female parent

  $\forall m,c\ Mother(c) = m \Leftrightarrow (Female(m) \wedge Parent(m,c))$
- "Sibling" is symmetric

  $\forall x,y\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$

Using FOL

The set domain:

- The only sets are the empty set and those made by adjoining something to a set:
  - $\forall s\ Set(s) \Leftrightarrow (s = \{\}) \vee (\exists x,s_2\ Set(s_2) \wedge s = \{x|s_2\})$
- The empty set has no elements ajoined into it
  - $\neg\exists x,s\ \{x|s\} = \{\}$
- Two sets are equal if and only if it is a member of both sets.
  - $\forall s_1,s_2\ (s_1 = s_2) \Leftrightarrow (s_1 \subseteq s_2 \wedge s_2 \subseteq s_1)$

Interacting with FOL KBs

- Suppose a wumpus-world agent is using an FOL KB and perceives a glitter and a breeze (but no smell) at *t=5*:

  `Tell`(KB,Percept([Glitter,Breeze,None],5))
  `Ask`(KB,∃a BestAction(a,5))

- I.e., does the KB entail some best action at *t=5*?
- Answer: *Yes*, {*a/Grab*} ← substitution (binding list)
- E.g,

  Smarter(Hillary,Bill)
  Smarter(x,y)
  σ = {x/Hillary,y/Bill}

- `Ask`(KB,S) returns some/all σ such that KB ⊨ σ

Knowledge base for the wumpus world

- Perception
  - $\forall t,s,b\ Percept([s,b,Glitter],t) \Rightarrow Glitter(t)$
- Reflex
  - $\forall t\ Glitter(t) \Rightarrow BestAction(Grab,t)$

Deducing hidden properties

- $\forall x,y,a,b\ \mathit{Adjacent}([x,y],[a,b]) \Leftrightarrow [a,b] \in \{[x+1,y], [x-1,y],[x,y+1],[x,y-1]\}$

Properties of squares:

- $\forall s,t\ \mathit{At}(Agent,s,t) \wedge Breeze(t) \Rightarrow Breezy(s)$

Squares are breezy near a pit:

- Diagnostic rule---infer cause from effect

  $\forall s\ Breezy(s) \Rightarrow \exists r\ Adjacent(r,s) \wedge Pit(r)$
- Causal rule---infer effect from cause

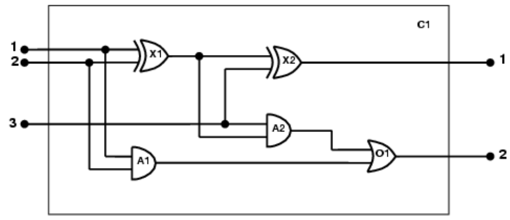
  $\forall r\ Pit(r) \Rightarrow [\forall s\ Adjacent(r,s) \Rightarrow Breezy(s)]$

Knowledge engineering in FOL

1. Identify the task
2. Assemble the relevant knowledge
3. Decide on a vocabulary of predicates, functions, and constants
4. Encode general knowledge about the domain
5. Encode a description of the specific problem instance
6. Pose queries to the inference procedure and get answers
7. Debug the knowledge base

The electronic circuits domain

One-bit full adder



The electronic circuits domain

1. Identify the task
   – Does the circuit actually add properly? (circuit verification)
2. Assemble the relevant knowledge
   – Composed of wires and gates; Types of gates (AND, OR, XOR, NOT)
   – Irrelevant: size, shape, color, cost of gates
3. Decide on a vocabulary
   – Alternatives:
     $Type(X_1) = XOR$
     $Type(X_1, XOR)$
     $XOR(X_1)$

The electronic circuits domain

4. Encode general knowledge of the domain
   – $\forall t_1,t_2\ Connected(t_1, t_2) \Rightarrow Signal(t_1) = Signal(t_2)$
   – $\forall t\ Signal(t) = 1 \vee Signal(t) = 0$
   – $1 \neq 0$
   – $\forall t_1,t_2\ Connected(t_1, t_2) \Rightarrow Connected(t_2, t_1)$
   – $\forall g\ Type(g) = OR \Rightarrow Signal(Out(1,g)) = 1 \Leftrightarrow \exists n\ Signal(In(n,g)) = 1$
   – $\forall g\ Type(g) = AND \Rightarrow Signal(Out(1,g)) = 0 \Leftrightarrow \exists n\ Signal(In(n,g)) = 0$
   – $\forall g\ Type(g) = XOR \Rightarrow Signal(Out(1,g)) = 1 \Leftrightarrow Signal(In(1,g)) \neq Signal(In(2,g))$
   – $\forall g\ Type(g) = NOT \Rightarrow Signal(Out(1,g)) \neq Signal(In(1,g))$

The electronic circuits domain

5. Encode the specific problem instance

$Type(X_1) = XOR$ $Type(X_2) = XOR$
$Type(A_1) = AND$ $Type(A_2) = AND$
$Type(O_1) = OR$

$Connected(Out(1,X_1),In(1,X_2))$ $Connected(In(1,C_1),In(1,X_1))$
$Connected(Out(1,X_1),In(2,A_2))$ $Connected(In(1,C_1),In(1,A_1))$
$Connected(Out(1,A_2),In(1,O_1))$ $Connected(In(2,C_1),In(2,X_1))$
$Connected(Out(1,A_1),In(2,O_1))$ $Connected(In(2,C_1),In(2,A_1))$
$Connected(Out(1,X_2),Out(1,C_1))$ $Connected(In(3,C_1),In(2,X_2))$
$Connected(Out(1,O_1),Out(2,C_1))$ $Connected(In(3,C_1),In(1,A_2))$

The electronic circuits domain

6. Pose queries to the inference procedure

What are the possible sets of values of all the terminals for the adder circuit?

$\exists i_1,i_2,i_3,o_1,o_2$ $Signal(In(1,C\_1)) = i_1 \wedge Signal(In(2,C_1)) = i_2 \wedge Signal(In(3,C_1)) = i_3 \wedge Signal(Out(1,C_1)) = o_1 \wedge Signal(Out(2,C_1)) = o_2$

7. Debug the knowledge base

May have omitted assertions like $1 \neq 0$

Summary

- First-order logic:
  - objects and relations are semantic primitives
  - syntax: constants, functions, predicates, equality, quantifiers
- Increased expressive power