Bayesian Networks

4. Exact Inference / Variable Elimination

Lars Schmidt-Thieme

Information Systems and Machine Learning Lab (ISMLL)
Institute of Computer Science
University of Hildesheim
http://www.ismll.uni-hildesheim.de

1. Inference in Probabilistic Networks

2. Variable elimination

studfarm example

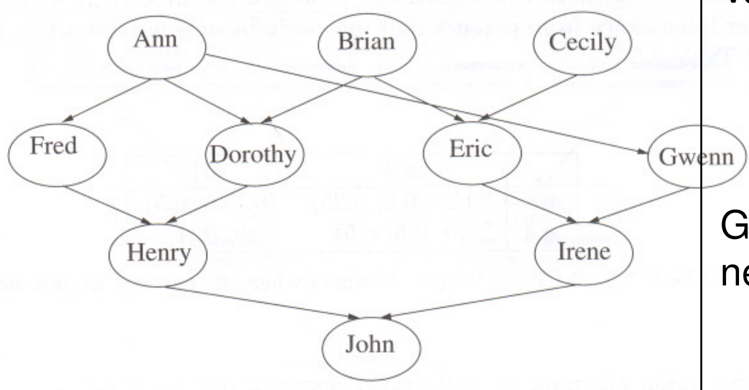


Figure 1: Genealogical structure for the horses in the studfarm example [Jen01, p. 47].

	aa	aA	AA
aa	(1, 0, 0)	(0.5, 0.5, 0)	(0, 1, 0)
aA	(0.5, 0.5, 0)	(0.25, 0.5, 0.25)	(0, 0.5, 0.5)
AA	(0, 1, 0)	(0, 0.5, 0.5)	(0, 0, 1)

Figure 2: p(Child | Father, Mother) for genetic inheritance. The numbers are the probabilities for (aa, aA, AA) [Jen01, p. 47].

Variable *disease* with three states:

pure (aa) carrier (aA) sick (AA)

Genalogic graph becomes bayesian network if

(i) each non-root vertex has conditional probability distribution

$$p(\text{child}|\text{father}, \text{mother})$$

as given in fig. 2,

(ii) each root vertex has probability distribution

$$p(aa) = .99, p(aA) = .01, p(AA) = .0$$

studfarm example

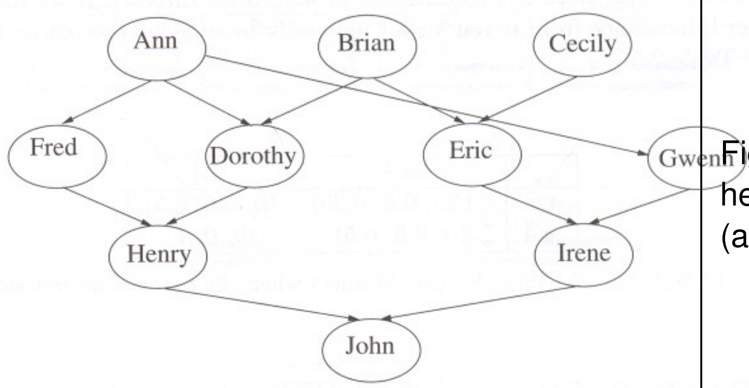


Figure 3: Genealogical structure for the horses in the studfarm example [Jen01, p. 47].

	aa	aA	AA
aa	(1, 0, 0)	(0.5, 0.5, 0)	(0, 1, 0)
aA	(0.5, 0.5, 0)	(0.25, 0.5, 0.25)	(0, 0.5, 0.5)
AA	(0, 1, 0)	(0, 0.5, 0.5)	(0, 0, 1)

Figure 4: p(Child | Father, Mother) for genetic inheritance. The numbers are the probabilities for (aa, aA, AA) [Jen01, p. 47].

father	aa			aA			AA		
mother	aa	aA	AA	aa	aA	AA	aa	aA	AA
aa	1	.5	0	.5	.25	0	0	0	0
aA	0	.5	1	.5	.5	.5	1	.5	0
AA	0	0	0	0	.25	.5	0	.5	1

father	aa		aA	
mother	aa	aA	aa	aA
aa	1	.5	.5	.25
aA	0	.5	.5	.5
AA	0	0	0	.25

father	aa		aA	
mother	aa	aA	aa	aA
aa	1	.5	.5	$\frac{1}{3}$
aA	0	.5	.5	$\frac{2}{3}$

Figure 5: p(child | father, mother) in general (left), if father and mother cannot be sick (middle), and if child cannot be sick either (right).

studfarm example / "forward inference"

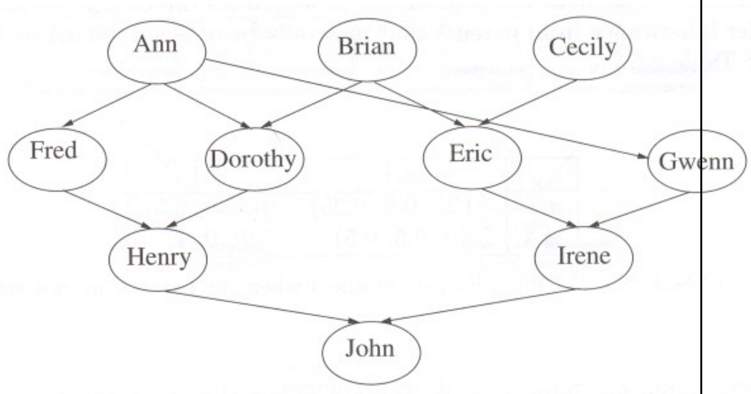


Figure 6: Genealogical structure for the horses in the studfarm example [Jen01, p. 47].

father	aa		aA	
mother	aa	aA	aa	aA
aa	1	.5	.5	.25
aA	0	.5	.5	.5
AA	0	0	0	.25

Figure 7: p(child | father, mother) if father and mother cannot be sick.

$$p(aa) = 0.99 \cdot 0.99 + 2 \cdot \frac{1}{2} \cdot 0.99 \cdot 0.01 + \frac{1}{4} \cdot 0.01 \cdot 0.01 = 0.990025$$

$$p(aA) = + 2 \cdot \frac{1}{2} \cdot 0.99 \cdot 0.01 + \frac{1}{2} \cdot 0.01 \cdot 0.01 = 0.00995$$

$$p(AA) = + \frac{1}{4} \cdot 0.01 \cdot 0.01 = 0.000025$$

studfarm example / "forward inference"

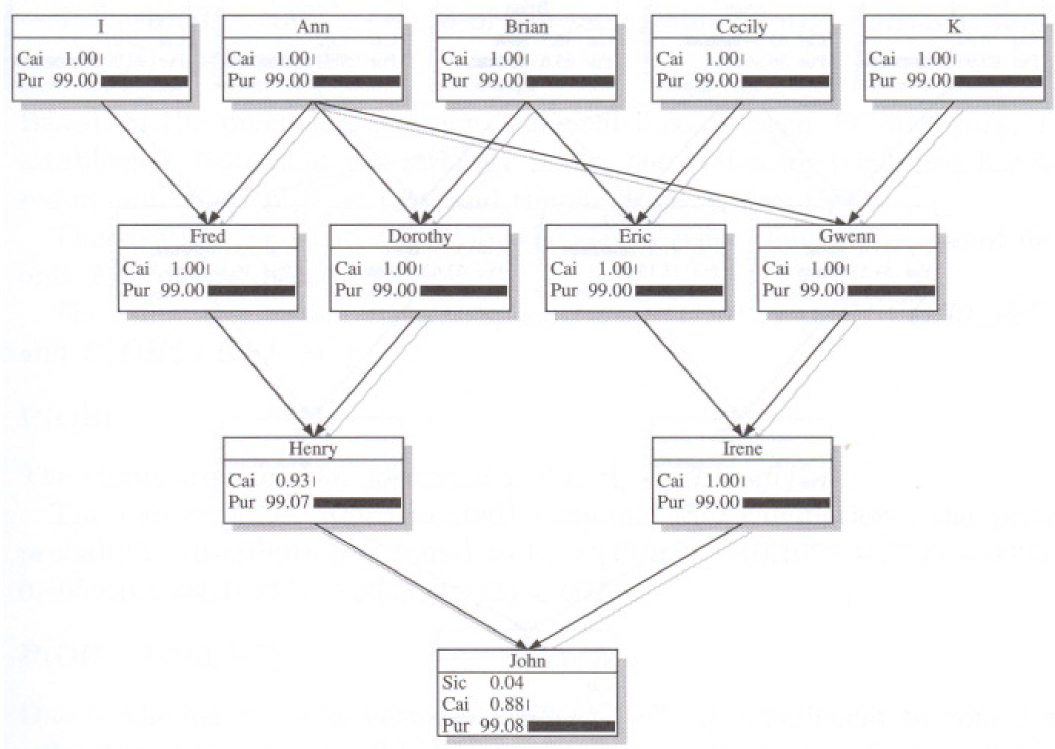


Figure 8: Probabilities without evidence. [Jen01, p. 49]

studfarm example / "backward inference"

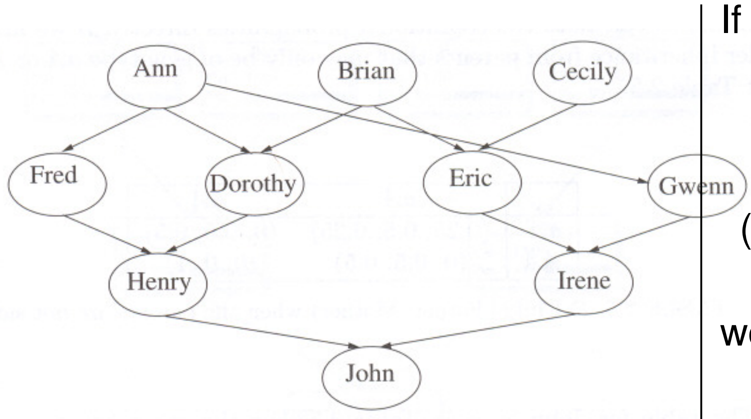


Figure 9: Genealogical structure for the horses in the studfarm example [Jen01, p. 47].

father	aa		aA	
mother	aa	aA	aa	aA
aa	1	.5	.5	.25
aA	0	.5	.5	.5
AA	0	0	0	.25

Figure 10: p(child | father, mother) if father and mother cannot be sick.

If we know, that

(i) all horses but John are not sick and

(ii) John is sick (AA),

we can infer that

(iii) Henry and Irene are carrier (aA) with $p = 1$.

If only Fred, Dorothy, Erik, and Gwen existed, we could further infer that for each of them

$$p(aa) = \frac{1}{3}, \quad p(aA) = \frac{2}{3}$$

studfarm example / "backward inference"

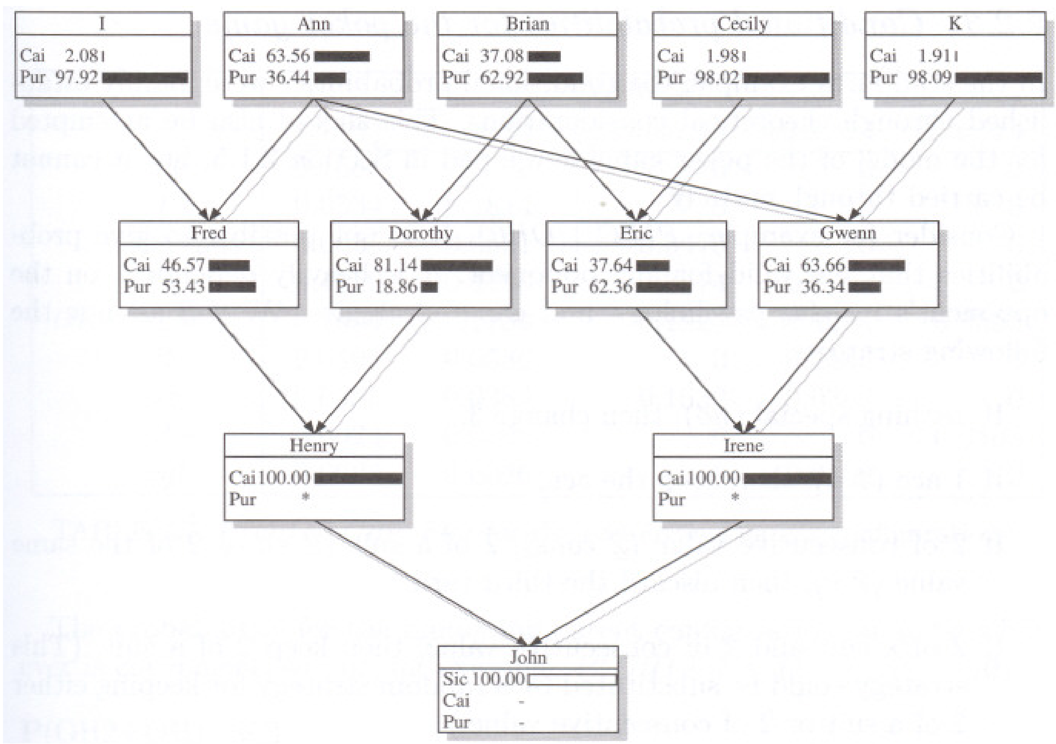


Figure 11: Probabilities given evidence that John is sick (AA). [Jen01, p. 49]

Evidence

Definition 1. Let V be a set of variables. The set

$$\mathcal{E} := \{E \subseteq \bigcup_{v\in V} \{v\} \times \operatorname{dom}(v) \,|\, \forall (v,c),(v,c') \in E : c = c'\}$$

is called **space of evidence of** V.
An element $E \in \mathcal{E}$ is called **evidence of** V. We call

$$\operatorname{dom}(E) := \{v \in V \,|\, \exists c \in \operatorname{dom}(v) : (v,c) \in E\}$$

the **set of evidential variables** and for each evidential variable $v \in \operatorname{dom}(E)$ we call the unique $E_v := c \in \operatorname{dom}(v)$ with $(v,c) \in E$ its **(evidential) value**.
Evidence E corresponds to the probability distribution

Evidence is a setting of variables to specific values. "Fuzzy" or "uncertain evidence" that assigns probabilities to the different values of the variables, is not handled here.

$$\begin{aligned}\operatorname{epd}_E : \prod_{v\in \operatorname{dom}(E)} \operatorname{dom}(v) &\rightarrow \mathbb{R}_0^+ \\ (x)_{v\in\operatorname{dom}(E)} &\mapsto \begin{cases}1, & \text{if } \forall v : (v,x) \in E \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

Evidence / example

Example 1. Let $V := \{A, B, C, D\}$ and

$$\begin{aligned}\operatorname{dom}(A) := \operatorname{dom}(B) &:= \{0,1\}, \\ \operatorname{dom}(C) &:= \{0,1,2\} \text{ and} \\ \operatorname{dom}(A) &:= \{0,1,2,3\}.\end{aligned}$$

Then

$$E := \{(A,1),(C,2)\}$$

is an evidence with the evidential variables A and C. The evidential variable A has value 1, the variable C value 2.

The probability distribution corresponding to E is

$$\operatorname{epd}_E(A=1, C=2) = 1$$

and

$$\operatorname{epd}_E(A=a, C=c) = 0$$

for all other values a of A or c of C.

Entering evidence

Let V be a set of variables and q be a potential on a subset of V. Let E be evidence of V.

We call

$$q_E : \prod_{v \in \mathrm{dom}(q)\setminus \mathrm{dom}(E)} \mathrm{dom}(v) \rightarrow \mathbb{R}_0^+$$
$$(x)_{v\in\mathrm{dom}(q)\setminus\mathrm{dom}(E)} \mapsto q(x, E)$$

with

$$(x, E)(v) := \begin{cases} x_v, & \text{if } v \in \mathrm{dom}(q) \setminus \mathrm{dom}(E) \\ E_v, & \text{if } v \in \mathrm{dom}(E) \end{cases}$$

the potential q given evidence E.

If q is a JPD, then q_E is the probability distribution on the non-evidential variables $\mathrm{dom}(q) \setminus \mathrm{dom}(E)$ for outcomes that conform to E (i.e., have value E_v for each variable $v \in \mathrm{dom}(E)$).

Warning: q_E should not be confused with the conditional probability distribution $q^{|\,\mathrm{dom}(E)}$. In sloppy notation for $E = \{(v_1, c_1), \ldots, (v_n, c_n)\}$:

$$q_E = q(x, v_1 = c1, \ldots, v_n = c_n)$$

and

$$q^{|\,\mathrm{dom}(E)} = q(x \,|\, v_1, \ldots, v_n)$$

Inferencing

Given a JPD p on a set of variables V and evidenve E on V.

We distinguish three types of inference targets:

(i) a single variable: For a given variable $v \in V$ **infering** v **based on** E **w.r.t.** p means to compute

$$p(v|E) = \frac{p(v, E)}{p(E)} \sim p(v, E)$$

or (more exactly) $(p_E)^{\downarrow v|\emptyset}$.

(ii) several variables separately: For a given set of variables $W \subseteq V$ **infering** W **separately based on** E **w.r.t.** p means to compute

$$p(v|E) = \frac{p(v, E)}{p(E)} \sim p(v, E), \quad \forall v \in W$$

or $(p_E)^{\downarrow v|\emptyset}$

(iii) joint distribution of several variabl
For a given set of variables $W \subseteq V$ **infering the marginal** W **based on** E **w.r.t.** p means to compute

$$p(W|E) = \frac{p(W, E)}{p(E)} \sim p(W, E)$$

or $(p_E)^{\downarrow W|\emptyset}$

Normalizing is necessary, as p_E in general is not a probability distribution, even if p is.

Inferencing / JPD as one large table

If p is given as one large table, infering the marginal W based on E means

(i) select the subtable indexed by E,

(ii) aggregate to W, i.e., sum over all variables $V \setminus \mathrm{dom}(E) \setminus W$,

(iii) normalize.

Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	220	220	25	25	95	95	10	10
N	4	9	5	12	31	76	50	113

Figure 12: JPD p given as one large table.

Pain	Y		N	
Weightloss	Y	N	Y	N
Adeno Y	220	25	95	10
N	4	5	31	50

Figure 13: Subtable for $E = \{(V, Y)\}$: distribution p_E before normalization.

If we observe the evidence $V = Y$, then

$$p(\text{adeno}{=}Y|V = Y) = \sum_{w,q} p(\text{adeno}{=}Y, W = w, P = q|V = Y)$$
$$= \frac{220 + 25 + 95 + 10}{224 + 30 + 126 + 60} = \frac{350}{440} = 0.80$$

Inferencing / JPD as product of potentials

If p is given as product of potentials, i.e.,

$$p := (\prod_{q \in Q} q)^{|\emptyset}$$

the problem becomes more interesting.

Naive approach: we reduce the problem to inference w.r.t. p as one large table by explicitly computing p and then doing inference as on the former slide, actually computing

$$(p_E)^{\downarrow W|\emptyset} = (((\prod_{q \in Q} q)^{|\emptyset})_E)^{\downarrow W|\emptyset}$$

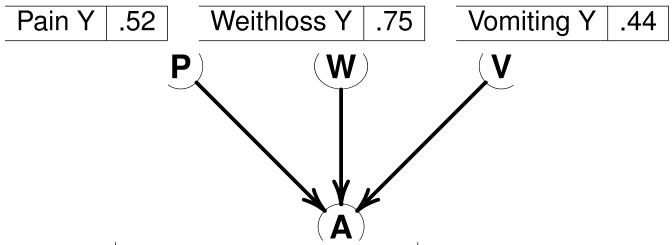
Naive approach$_2$: we

(i) enter evidence in the factors first, i.e., compute q_E, and then

(ii) compute p_E as product of the q_E's

$$(p_E)^{\downarrow W|\emptyset} = ((\prod_{q \in Q} q_E)^{\downarrow W|\emptyset})$$

product of potentials / naive approach



Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	.982	.961	.833	.676	.754	.556	.167	.081

Figure 14: Bayesian Network for adeno JPD.

Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	.169	.210	.048	.049	.119	.112	.009	.005
N	.003	.009	.010	.024	.039	.090	.044	.062

Figure 15: JPD of Bayesian Network for adeno JPD.

product of potentials / naive approach

Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	.169	.210	.048	.049	.119	.112	.009	.005
N	.003	.009	.010	.024	.039	.090	.044	.062

Figure 16: JPD p given as one large table.

Pain	Y		N	
Weightloss	Y	N	Y	N
Adeno Y	.169	.048	.119	.009
N	.003	.010	.039	.044

Figure 17: Subtable for $E = \{(V, Y)\}$: distribution p_E before normalization.

Adeno Y	.345
N	.096

Figure 18: Aggregate subtable for $E = \{(V, Y)\}$.

If we observe the evidence $V = Y$, then

$$p(\text{adeno=}Y|V = Y) = \sum_{w,q} p(\text{adeno=}Y, W = w, P = q|V = Y)$$

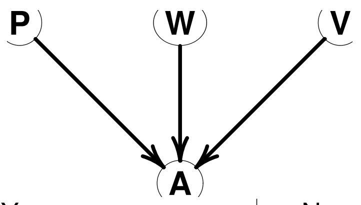
$$= \frac{.345}{.345 + .096} = 0.782$$

product of potentials / naive approach$_2$

Pain Y	.52

Weithloss Y	.75

Vomiting Y	~~.44~~ 1.0



Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	.982	.961	.833	.676	.754	.556	.167	.081

Figure 19: Bayesian Network for adeno JPD.

Pain	Y				N			
Weightloss	Y		N		Y		N	
Vomiting	Y	N	Y	N	Y	N	Y	N
Adeno Y	.384	0	.109	0	.270	0	.020	0
N	.007	0	.023	0	.089	0	.100	0

Figure 20: JPD of Bayesian Network for adeno JPD with evidenve $V = Y$ entered.

Overview of inference methods [Guo and Hsu 2001]

(i) exact inference:

- (a) Polytree algorithm
- (b) conditioning
- (c) clustering
- (d) arc reversal
- (e) variable elimination

(ii) approximate inference:

- (a) stochastic sampling
- (b) model simplification
- (c) search-based
- (d) loopy propagation

(iii) symbolic inference.

1. Inference in Probabilistic Networks

2. Variable elimination

Aggregating products

Doing inference using the naive approach$_2$,

$$(p_E)^{\downarrow W|\emptyset} = ((\prod_{q\in Q} q_E)^{\downarrow W|\emptyset})$$

we compute a large table as product of q_E and then aggregate to W.

Question: can we aggregate the factors and then multiply the aggregates?

$$(pq)^{\downarrow W} \stackrel{?}{=} p^{\downarrow W} q^{\downarrow W}$$

In general, this equation does not hold, as

$$(pq)^{\downarrow W}(x) = \sum_{y\in\prod_{X\in \mathrm{dom}(pq)\setminus W} \mathrm{dom}(X)} p(x,y)q(x,y)$$

but

$$(p^{\downarrow W} q^{\downarrow W})(x) = (\sum_{y\in\prod_{X\in \mathrm{dom}(p)\setminus W} \mathrm{dom}(X)} p(x,y))\cdot(\sum_{y\in\prod_{X\in \mathrm{dom}(q)\setminus W} \mathrm{dom}(X)} q(x,y))$$

But it is true for $\mathrm{dom}(p) \cap \mathrm{dom}(q) \subseteq W$, i.e., if p and q have no common variables except those in W.

Lemma 1. *Let p and q be two potentials on a subset of variables V. Let $W \subseteq V$ a subset of the variables.*
If $\mathrm{dom}(p) \cap \mathrm{dom}(q) \subseteq W$ then

$$(pq)^{\downarrow W} = p^{\downarrow W} q^{\downarrow W}$$

Variable elimination

We can make use of this observation for simplifying $(\prod_{q\in Q} q)^{\downarrow W}$:

(i) choose a variable $v \in V \setminus W$, clearly

$$(\prod_{q\in Q} q)^{\downarrow W} = ((\prod_{q\in Q} q)^{\downarrow cv})^{\downarrow W}$$

i.e., we can eliminate variable v first,

(ii) let

$$R := \{q \in Q \mid v \in \mathrm{dom}(q)\}$$

be all potentials which's domain contains v and

$$q' := \prod_{q\in R} q, \quad q_{\text{rest}} = \prod_{q\in Q\setminus R} q$$

(iii) Then

$$\mathrm{dom}(q') \cap \mathrm{dom}(q_{\text{rest}}) \subseteq V \setminus \{v\}$$

and thus

$$(\prod_{q\in Q} q)^{\downarrow W} = (q_{\text{rest}} \cdot q'^{\downarrow cv})^{\downarrow W}$$

i.e., we replace the potentials R by

$$q'^{\downarrow cv}$$

After this replacement, the variable v is eliminated from the potentials $Q' := Q \setminus R \cup \{q'^{\downarrow cv}\}$.

Variable elimination

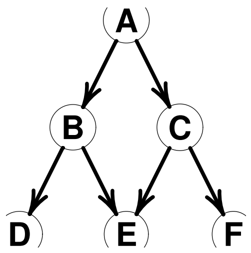
```
inference-varelim(Q : set of potentials, W : set of variables) :
while ⋃_{q∈Q} dom(q) \ W ≠ ∅ do
      choose v ∈ ⋃_{q∈Q} dom(q) \ W arbitrarily
      Q := eliminate-variable(Q, v)
od
return (∏_{q∈Q} q)^{|∅}

eliminate-variable(Q : set of potentials, v : variable) :
R := {q ∈ Q | v ∈ dom(q)}
q' := (∏_{q∈R} q)^{↓cv}
return Q \ R ∪ {q'}
```

Also known as **bucket elimination**.

Useful if the set W of variables to infer separately is small.

example

Example 2. Let $(G, (p_v)_{v \in V})$ be the following Bayesian network



The conditional probabilities are

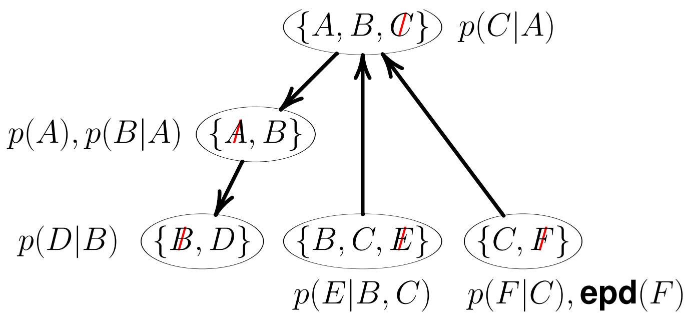
$$Q := \{p(A), p(B|A), p(C|A), p(D|B), \\ p(E|B,C), p(F|C)\}$$

We want to compute the marginal $p(D)$ given evidence on F. Thus we add $\text{epd}(F)$ to Q.

For the elimination sequence

$$F, E, C, A, B$$

the following steps have to be performed:

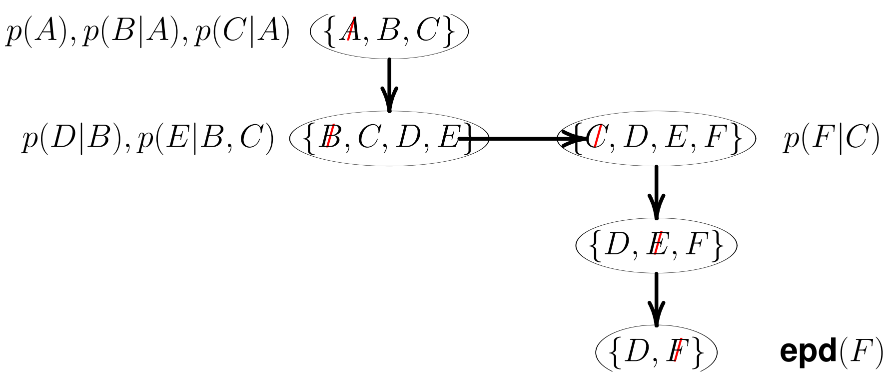


example

For the elimination sequence

$$A, B, C, E, F$$

the following steps have to be performed:



References

[Jen01] Finn V. Jensen. *Bayesian networks and decision graphs*. Springer, New York, 2001.