Computer Vision

1. Projective Geometry in 2D

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Outline

Outline

Topics of the Lecture

1. Simultaneous Localization and Mapping from Video (Visual SLAM)

2. Image Classification and Description

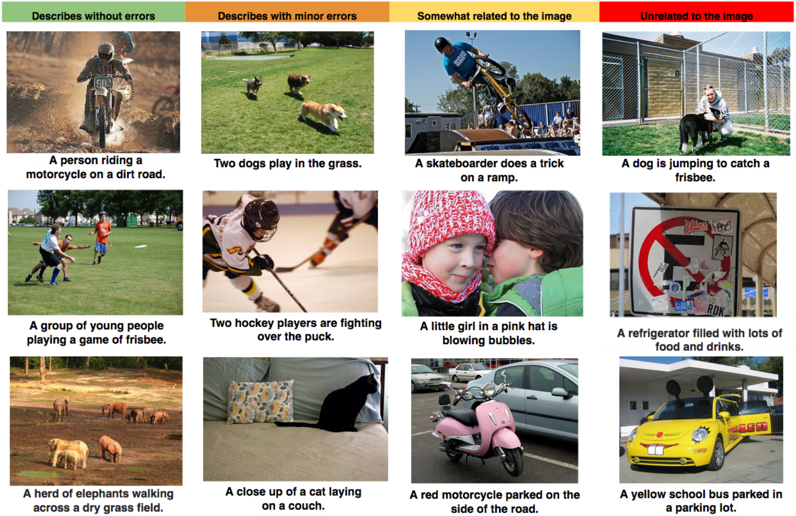
Simultaneous Localization and Mapping

[source https://www.youtube.com/watch?v=bDOnn0-4Nq8]

Simultaneous Localization and Mapping from Video

- SLAM usually employs laser range scanners (lidars).
- **Visual SLAM**: use video sensors (cameras).
- main parts required:
  1. Projective Geometry
  2. Point Correspondences
  3. Estimating Camera Positions (Localization)
  4. Triangulation (Mapping)

Image Classification and Description



[source: http://googleresearch.blogspot.de/2014/11/a-picture-is-worth-thousand-coherent.html]

Outline

Motivation

In Euclidean (planar) geometry, there are many exceptions, e.g.,

- most two lines intersect in exactly one point.
- but some two lines do not intersect.
  - parallel lines

Motivation

In Euclidean (planar) geometry, there are many exceptions, e.g.,

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- but some two lines do not intersect.
  - parallel lines

Idea:

- add **ideal points**, one for each set of parallel lines / direction
- define these points as intersection of any two parallel lines
- now any two lines intersect in exactly one point
  - either in a finite or in an ideal point

Homogeneous Coordinates: Points

Inhomogeneous coordinates:

$$x \in \mathbb{R}^2$$

Homogeneous coordinates:

$$x \in \mathbb{P}^2 := \mathbb{R}^3 / \equiv$$

$$x \equiv y :\Longleftrightarrow \exists s \in \mathbb{R} \setminus \{0\} : sx = y, \quad x, y \in \mathbb{R}^3$$

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Example:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix} \text{ represent the same point in } \mathbb{P}^2$$

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \text{ represent a different point in } \mathbb{P}^2$$

Homogeneous Coordinates: Points

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finite points: $\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} =: \iota(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix})$

ideal points: $\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

Homogeneous Coordinates: Lines

Inhomogeneous coordinates:

$$a \in \mathbb{R}^3 : \ell_a := \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid a_1 x_1 + a_2 x_2 + a_3 = 0 \}$$

- $a_1 \neq 0$ or $a_2 \neq 0$ (or both $a_1, a_2 \neq 0$).
- $sa = (sa_1, sa_2, sa_3)^T$ encodes the same line as a (any $s \in \mathbb{R}, s \neq 0$).

Note: $\kappa : \mathbb{R}^3 \to \mathbb{P}^2, a \mapsto [a] := \{a' \in \mathbb{R}^3 \mid a' \equiv a\}$.

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Homogeneous coordinates:

$$a \in \mathbb{P}^2 : \ell_a := \{ x \in \mathbb{P}^2 \mid a^T x = a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \}$$

- contains all finite points of $a' \in \kappa^{-1}(a)$: $\ell_{\kappa(a')} \supsetneq \iota(\ell_{a'})$
- and the ideal point $(a_2, -a_1, 0)^T$.
  - intersection of parallel lines (same a_1, a_2, different a_3)

Note: $\kappa : \mathbb{R}^3 \to \mathbb{P}^2, a \mapsto [a] := \{ a' \in \mathbb{R}^3 \mid a' \equiv a \}$.

A point on a line

A point x lies on line a iff $x^T a = 0$.

Intersection of two lines

Lines a and b intersect in $a \times b := \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

Proof:

$$a^T(a \times b) = a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0$$
$$b^T(a \times b) = \ldots = 0$$

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$$b^T(a \times b) = \ldots = 0$$

Example:

$$x = 1 : a = (-1, 0, 1)^T$$
$$y = 1 : b = (0, -1, 1)^T$$
$$a \times b = (1, 1, 1)^T$$

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$$b^T(a \times b) = \ldots = 0$$

Esp. for parallel lines: $b_1 = a_1, b_2 = a_2, b_3 \neq a_3$:

$$a \times b \equiv \begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix}$$

Line joining points

The line through x and y is $x \times y$.

Proof: exactly the same as previous slide.

Line joining points

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Example:

$$
\begin{aligned}
x &= (-1, 0, 1)^T \\
y &= (0, -1, 1)^T \\
x \times y &= (1, 1, 1)^T
\end{aligned}
$$

Line at infinity

All ideal points form a line:

$$l_\infty := (0, 0, 1)^T \quad \textbf{line at infinity}$$

Proof:

for any ideal point $x = (x_1, x_2, 0)^T$: $x^T l_\infty = 0$.

for any finite (real-valued) point $x = (x_1, x_2, 1)$: $x^T l_\infty = 1 \neq 0$.

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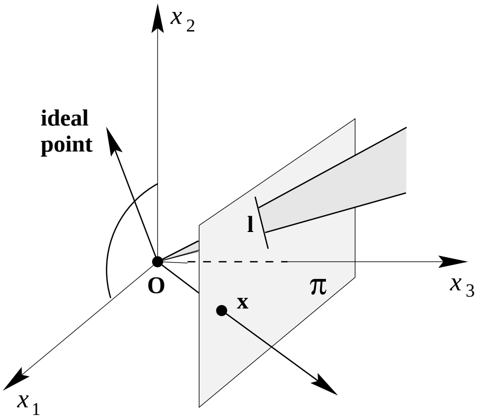
Furthermore:

- This is the only line in $\mathbb{P}^2$ **not** corresponding to an Euclidean line.
- Two parallel lines meet at the line at infinity.

A model for the projective plane



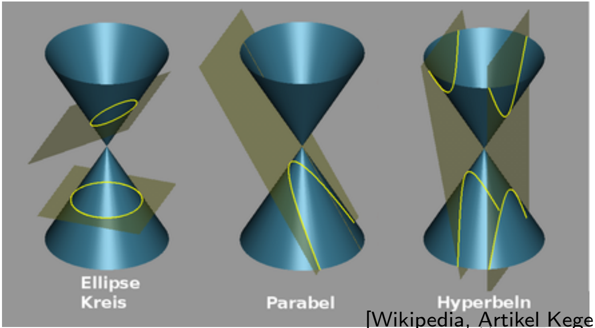
- points correspond to rays (lines through the origin)
- lines correspond to planes through the origin.

[HZ04, p. 29]

Conics

- A **conic section** (or just **conic**) is a curve one gets as intersection of a cone and a plane
  - ellipsis, parabola, hyperbola
- Corresponds to a curve of degree 2:
  Heterogeneous coordinates:

$$a \in \mathbb{R}^6 : \mathbf{C}_a := \{x \in \mathbb{R}^2 \mid a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1 + a_5 x_2 + a_6 = 0\}$$



[Wikipedia, Artikel Kegelschnitt]

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$$a \in \mathbb{P}^5 : \mathbf{C}_a := \{x \in \mathbb{P}^2 \mid a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1 x_3 + a_5 x_2 x_3 + a_6 x_3^2 = 0\}$$

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  Homogeneous coordinates:

$$\begin{aligned} a \in \mathbb{P}^5 : \mathbf{C}_a := \{x \in \mathbb{P}^2 \mid\ & a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 \\ & + a_4 x_1 x_3 + a_5 x_2 x_3 + a_6 x_3^2 = 0\} \\ = \{x \in \mathbb{P}^2 \mid x^T C x = 0\},\ & C := \begin{pmatrix} a_1 & a_2/2 & a_4/2 \\ a_2/2 & a_3 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix} \end{aligned}$$

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  Homogeneous coordinates:

$$C \in \mathrm{Sym}(\mathbb{P}^{3\times 3}) : \mathbf{C}_C := \{x \in \mathbb{P}^2 \mid x^T C x = 0\}$$

A conic joining 5 points

- Let $x^1, \ldots, x^5 \in \mathbb{P}^2$ be 5 points
  - in general position (i.e., never more than 2 on the same line)
- Conic parameters a have to fulfil the following system of linear equations:

$$\begin{pmatrix} x^1_1x^1_1 & x^1_1x^1_2 & x^1_2x^1_2 & x^1_1x^1_3 & x^1_2x^1_3 & x^1_3x^1_3 \\ x^2_1x^2_1 & x^2_1x^2_2 & x^2_2x^2_2 & x^2_1x^2_3 & x^2_2x^2_3 & x^2_3x^2_3 \\ x^3_1x^3_1 & x^3_1x^3_2 & x^3_2x^3_2 & x^3_1x^3_3 & x^3_2x^3_3 & x^3_3x^3_3 \\ x^4_1x^4_1 & x^4_1x^4_2 & x^4_2x^4_2 & x^4_1x^4_3 & x^4_2x^4_3 & x^4_3x^4_3 \\ x^5_1x^5_1 & x^5_1x^5_2 & x^5_2x^5_2 & x^5_1x^5_3 & x^5_2x^4_3 & x^5_3x^5_3 \end{pmatrix} a = 0$$

Degenerate Conics

Conic C **degenerate**: C does not have full rank.

Example: two lines $C := ab^T + ba^T$ (rank 2).

- contains lines a and b.
  proof: for points x on line a: $x^T a = 0$.
  $\rightsquigarrow x$ also on C: $x^T C x = x^T ab^T x + x^T ba^T x = 0$.

Conic tangent lines

The tangent line to a conic C at a point x is Cx.

Proof:

x lies on Cx: $x^T Cx = 0$.
If there is another common point y: $y^T Cy = 0$ and $y^T Cx = 0$.
$\rightsquigarrow$ $x + \alpha y$ is common for all α, i.e., the whole line.
$\rightsquigarrow$ C is degenerate (or there is no such y).

Outline

Projectivity

A map $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is called **projectivity**, if

1. it is invertible and
2. it preserves lines,
   i.e., whenever x, y, z are on a line, so are $h(x), h(y), h(z)$.

Equivalently, $h(x) := Hx$ for a non-singular $H \in \mathbb{P}^{3\times 3}$.

Note: $H^{-T} := (H^{-1})^T$.

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Equivalently, $h(x) := Hx$ for a non-singular $H \in \mathbb{P}^{3\times 3}$.

Proof:
Any map $h(x) := Hx$ is a projectivity:
Let x be a point on line a: $a^T x = 0$.
Then point Hx is on line $H^{-T}a$: $(H^{-1}a)^T Hx = a^T H^{-1} Hx = a^T x = 0$.

Any projectivity h is of type $h(x) = Hx$: more difficult to show.

Note: $H^{-T} := (H^{-1})^T$.

Transformation of Lines and Conics

The image of a line a under projectivity H is the line $H^{-T}a$:

$$H(l_a) = l_{H^{-T}a}$$

Proof:

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The image of a conic C under projectivity H is the conic $H^{-T}CH^{-1}$:

$$H(\mathbf{C}_C) = \mathbf{C}_{H^{-T}CH^{-1}}$$

Proof:

Let x be a point on conic C: $x^T Cx = 0$.
Then point Hx is on conic $H^{-T}CH^{-1}$: $x^T H^T H^{-T} CH^{-1} H^{-1} x = 0$

A Hierarchy of Transformations

- The projective transformations form a group (**projective linear group**:

$$\mathrm{PL}_n := \mathrm{GL}_n / \equiv = \{H \in \mathbb{P}^{3\times 3} \mid H \text{ invertible}\}$$

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- There are several subgroups:
  - **affine group**: last row is $(0, 0, 1)$
  - **Euclidean group**: additionally $H_{1:2,1:2}$ orthogonal
  - **oriented Euclidean group**: additionally $\det H = 1$

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- These subgroups can be described two ways:
  - structurally (as above)
  - by **invariants**: objects or sets of objects mapped to themselves

Isometries

$$\begin{pmatrix} x_1' \\ x_2' \\ 1 \end{pmatrix} = \begin{pmatrix} \epsilon \cos\theta & -\sin\theta & t_1 \\ \epsilon \sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} x$$

- **rotation matrix** R: $R^T R = RR^T = I$
- **translation vector** t.
- **orientation preserving** if $\epsilon = +1$ (equivalent to $\det R = +1$) ($\epsilon \in \{+1, -1\}$)

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Invariants:

- length, angle, area
- line at infinity l_∞

Similarity Transformations

$$\begin{pmatrix} x_1' \\ x_2' \\ 1 \end{pmatrix} = \begin{pmatrix} s\cos\theta & -s\sin\theta & t_1 \\ s\sin\theta & s\cos\theta & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} x$$

- **isotropic scaling** s.

Similarity Transformations

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Invariants:

- angle
- ratio of lengths, ratio of areas
- line at infinity l_∞

Affine Transformations

$$\begin{pmatrix} x_1' \\ x_2' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & t_1 \\ a_{2,1} & a_{2,2} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} x$$

- A non-singular

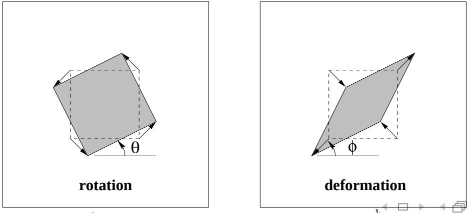
Affine Transformations

$$\begin{pmatrix} x'_1 \\ x'_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & t_1 \\ a_{2,1} & a_{2,2} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} x$$

- A non-singular, decompose via SVD:

$$A = R(\theta)R(-\phi)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R(\phi)$$

- **non-isotropic scaling** with axis ϕ



[HZ04, p. 40]

Affine Transformations

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Invariants:

- parallel lines
- ratio of lengths of parallel line segments
- ratio of areas
- line at infinity l_∞

Projective Transformations

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & t_1 \\ a_{2,1} & a_{2,2} & t_2 \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A & t \\ v^T & v_3 \end{pmatrix} x$$

- **v moves the line at infinity l_∞**

Projective Transformations

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- **v moves the line at infinity l_∞**

Invariants:

- ratio of ratios of lengths of parallel line segments (**cross ratio**)

Similary, Affine & Projective Transformations / Example

a

b

c

	a) similarity
circles	circles
squares	squares
parallel lines	parallel
orthogonal linnes	orthogonal

[HZ04, p. 37]

Similary, Affine & Projective Transformations / Example

a b c

	a) similarity	b) affine
circles	circles	ellipsis
squares	squares	diamond
parallel lines	parallel	parallel
orthogonal linnes	orthogonal	non-orthogonal

[HZ04, p. 37]

Similary, Affine & Projective Transformations / Example

a b c

	a) similarity	b) affine	c) projective
circles	circles	ellipsis	conic
squares	squares	diamond	quadrangle
parallel lines	parallel	parallel	converging
orthogonal linnes	orthogonal	non-orthogonal	non-orthogonal

[HZ04, p. 37]

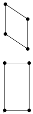
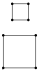
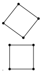
Projective Transformations / Decomposition

$$\begin{pmatrix} A & t \\ v^T & v_3 \end{pmatrix} = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix}$$

$$A = sRK + tv^T$$

- K upper triangular matrix with $\det K = 1$
- valid for $v_3 \neq 0$
- unique if s is chosen $s > 0$

Summary of Projective Transformations

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

[HZ04, p. 44]

Outline

Recovery of Affine and Metric Properties

Decomposition of general projective transformation:

$$\begin{pmatrix} A & t \\ v^T & v_3 \end{pmatrix} = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix}$$

1. undo proper projective transformation (**affine rectification**):
   - then original and image differ only by an affine transformation
   - $\rightsquigarrow$ measure **affine properties** of the original in the image (= properties invariant under affine transformations)
     - parallel lines, ratio of lengths on parallel lines

Recovery of Affine and Metric Properties

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1. undo proper projective transformation (**affine rectification**):
   - then original and image differ only by an affine transformation
   - $\rightsquigarrow$ measure **affine properties** of the original in the image (= properties invariant under affine transformations)
     - parallel lines, ratio of lengths on parallel lines
2. undo proper affine transformation (**metric rectification**):
   - then original and image differ only by a similarity transformation
   - $\rightsquigarrow$ measure **metric properties** of the original in the image (= properties invariant under similarity transformations)
     - angles, ratio of lengths

Recovery of Affine Properties

Undo proper projective transformation:

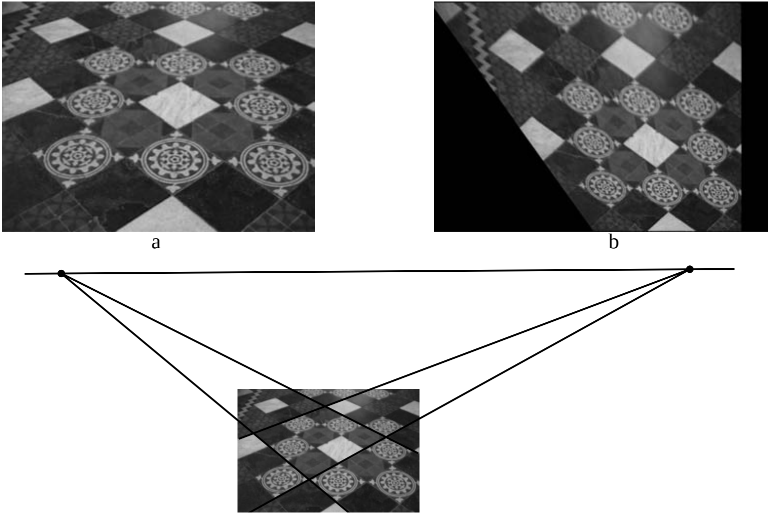
$$\begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

$$l_\infty := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -v/v_3 \\ 1/v_3 \end{pmatrix} = \frac{1}{v_3} \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix}$$

- maps line at infinity to finite line $(v_1, v_2, 1)^T$
- to undo:
  - locate image $(v_1, v_2, 1)^T$ of line at infinty
  - undo by applying the inverse $H^{-1} = \begin{pmatrix} I & 0 \\ -v^T/v_3 & 1/v_3 \end{pmatrix}$

Note: Lines transform by H^{-T}: $\begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix}^{-T} = \begin{pmatrix} I & -v/v_3 \\ 0 & 1/v_3 \end{pmatrix}$

Recovery of Affine Properties / Example



[HZ04, p. 50]

Now we can measure area ratios !

Recovery of Affine Properties / Algorithm

procedure $\textsc{Rectify-Affine-Two-Parallels}(a^1, a^2, b^1, b^2 \in \mathbb{P}^2)$

$s^1 := a^1 \times a^2$ ▷ compute intersection of parallels a^1, a^2

$s^2 := b^1 \times b^2$ ▷ compute intersection of parallels b^1, b^2

$l_\infty := s^1 \times s^2$ ▷ compute image of line at infinity

$H^{-1} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{\infty,1}/l_{\infty,3} & -l_{\infty,2}/l_{\infty,3} & 1/l_{\infty,3} \end{pmatrix}$ ▷ compute inverse

return H^{-1}

Outline

Circular Points

A conic

$$C := \begin{pmatrix} a_1 & a_2/2 & a_4/2 \\ a_2/2 & a_3 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & a_4/2 \\ 0 & a_1 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix}$$

is a circle if $a_1 = a_3$ and $a_2 = 0$.

Circular Points

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$$C := \begin{pmatrix} a_1 & a_2/2 & a_4/2 \\ a_2/2 & a_3 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & a_4/2 \\ 0 & a_1 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix}$$

is a circle if $a_1 = a_3$ and $a_2 = 0$.

Ideal points $x = (x_1, x_2, 0)^T$ on a circle:

$$x^T C x = a_1 x_1^2 + a_1 x_2^2 = 0$$

are exactly the **circular points**:

$$I := \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad J := \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

Line Conics

$C \in \mathrm{Sym}(\mathbb{P}^{3\times 3})$ defines a **point conic** via

$$\mathbf{C}_C := \{x \in \mathbb{P}^2 \mid x^T C x = 0\}$$

It also can be used to define a **line conic** / **dual conic**:

$$\mathbf{C}^*_C := \{a \in \mathbb{P}^2 \mid a^T C a = 0\}$$

(where a denotes a line)

Adjugate of a Matrix

For a square matrix $A \in \mathbb{R}^{n\times n}$,

$$A^* \in \mathbb{R}^{n\times n} \text{ with } A^*_{i,j} := (-1)^{i+j} \det A_{-j,-i}$$

is called its **adjugate** A^*.

It holds:

- for any A: $A^*A = AA^* = (\det A)I$
- A^* is continuous in A.
- if A is invertible, the adjoint is the scaled inverse: $A^* = (\det A)A^{-1}$
- if A is not invertible, the adjoint nullifies A: $A^*A = AA^* = 0$
- the adjugate is the transposed of the cofactor matrix.

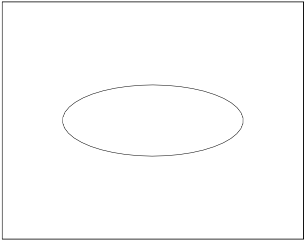
Note: $A_{-j,-i}$ denotes the matrix A with row j and column i removed.
The adjugate is also called **adjoint**.

Dual Conic

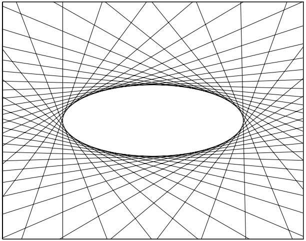
For any point conic $C \in \mathrm{Sym}(\mathbb{P}^{3\times 3})$, the set of tangent lines

- forms a line conic,
- parametrized by the adjugate C^*:

$$\{a \in \mathbb{P}^2 \mid a \text{ tangent to } C\} = \mathbf{C}^*_{C^*}$$



a



b

[HZ04, p. 32]

Dual Conic to the Circular Points

Dual conic to the circular points (degenerate):

$$C^*_\infty := IJ^T + JI^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- contains exactly all lines through the circular points I or J.
- transforms as HC^*H^T: $H(\mathbf{C}^*_{C^*}) = \mathbf{C}^*_{HC^*H^T}$.
- fixed under projectivity H iff H is a similarity.
- 4 dof (general C has 5, minus 1 due to $\det C = 0$)
- l_∞ is the null vector of C^*_∞.

Angels in the Projective Plane

Angels are defined as:

$$\cos\theta(a,b) := \frac{a^T C^*_\infty b}{\sqrt{(a^T C^*_\infty a)\,(b^T C^*_\infty b)}}, \quad a, b \in \mathbb{P}^2$$

- for the canonical C^*_∞, conincides with the Euclidean definition:

$$\cos\theta(a,b) := \frac{a^T b}{\sqrt{(a^T a)\,(b^T b)}}, \quad a, b \in \mathbb{R}^2$$

- stays invariant under projective transformation:

$$a' = H^{-T} a, \quad b' = H^{-T} b, \quad {C^*_\infty}' = H C^*_\infty H^T$$
$$a'^T {C^*_\infty}' b' = a^T H^{-1} H C^*_\infty H^T H^{-T} b = a^T C^*_\infty b$$

Outline

Recovery of Metric Properties

- assume there is no pure projective transformation (i.e., affine rectification already done).
- need only to find pure affine transformation:

$$H_a := \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix}, \quad \text{with } K \text{ upper triangular}$$

- under H_a we get $C_\infty^{*}{}'$ as

$$C_\infty^{*}{}' := H_a C_\infty^* H_a^T = \begin{pmatrix} KK^T & 0 \\ 0^T & 0 \end{pmatrix}$$

1. find symmetric matrix $S := KK^T$
2. find K via Cholesky decomposition of S

Recovery of Metric Properties (2/2)

- for two lines a', b' that are orthogonal in the original:

$$
\begin{aligned}
0 &= a'^T {C^*_\infty}' b' = {a'_{1:2}}^T S b_{1:2} \\
&= a'_1 S_{1,1} b'_1 + a'_1 S_{1,2} b'_2 + a'_2 S_{2,1} b'_1 + a'_2 S_{2,2} b'_2 \\
&= a'_1 b'_1 S_{1,1} + (a'_1 b'_2 + a'_2 b'_1) S_{1,2} + a'_2 b'_2 S_{2,2} \\
&= (a'_1 b'_1, a'_1 b'_2 + a'_2 b'_1, a'_2 b'_2)(S_{1,1}, S_{1,2}, S_{2,2})^T
\end{aligned}
$$

  we get 1 linear constraint in $s := (S_{1,1}, S_{1,2}, S_{2,2})^T$. .

- for two pairs of lines that are orthogonal in the original we get 2 linear constraints for 3 variables

$$
\begin{pmatrix} a'_1 b'_1 & a'_1 b'_2 + a'_2 b'_1 & a'_2 b'_2 \\ c'_1 d'_1 & c'_1 d'_2 + c'_2 d'_1 & c'_2 d'_2 \end{pmatrix} s
$$

  where $s \neq 0$ has to be identified only up to a factor.

Recovery of Metric Properties / Algorithm

procedure RECTIFY-METRIC-TWO-ORTHOGONALS$(a^1, a^2, b^1, b^2 \in \mathbb{P}^2)$

$$A := \begin{pmatrix} a^1_1 a^2_1 & a^1_1 a^2_2 + a^1_2 a^2_1 & a^1_2 a^2_2 \\ b^1_1 b^2_1 & b^1_1 b^2_2 + b^1_2 b^2_1 & b^1_2 b^2_2 \end{pmatrix}$$

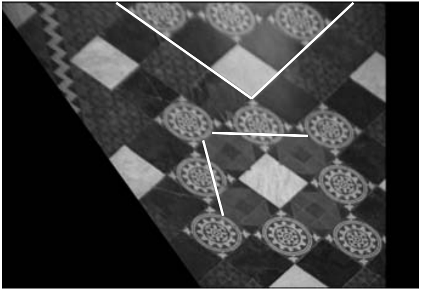
find $s \neq 0 : As = 0$ ▷ find $C^*_\infty := \begin{pmatrix} s_1 & s_2 & 0 \\ s_2 & s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$K := \text{cholesky}(\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix})$ ▷ find $H := \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix}$

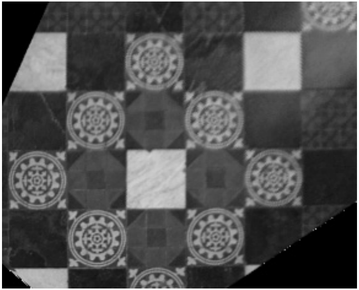
$$H^{-1} := \begin{pmatrix} 1/K_{1,1} & -1/(K_{1,2}K_{2,2}) & 0 \\ 0 & 1/K_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ▷ compute inverse

return H^{-1}

Recovery of Metric Properties / Example



a



b

a) affine rectified image

b) metric rectified image

Now we can measure angles and length ratios !

[HZ04, p. 57]

Outline

Exercises and Tutorials

- There will be a weekly sheet with 4 exercises
  handed out **each Tuesday** in the lecture.
  1st sheet will be handed out Thu. 23.4. in the tutorial.

- Solutions to the exercises can be
  submitted until **next Tuesday noon**
  1st sheet is due Tue. 28.4.

- Exercises will be corrected.

- Tutorials **each Thursday 2pm–4pm**,
  1st tutorial at Thur. 23.4.

- Successful participation in the tutorial gives up to 10% bonus points for the exam.

Exam and Credit Points

- There will be a written exam at end of term (2h, 4 problems).
- The course gives 6 ECTS (2+2 SWS).
- The course can be used in
  - IMIT MSc. / Informatik / Gebiet KI & ML
  - Wirtschaftsinformatik MSc / Informatik / Gebiet KI & ML
  - as well as in both BSc programs.

Some Text Books

- Simon J. D. Prince (2012):
  Computer Vision: Models, Learning, and Inference,
  Cambridge University Press.

- Richard Szeliski (2011):
  Computer Vision, Algorithms and Applications,
  Springer.

- David A. Forsyth, Jean Ponce (^{2}2012, 2007):
  Computer Vision, A Modern Approach,
  Prentice Hall.

- Richard Hartley, Andrew Zisserman (2004):
  Multiple View Geometry in Computer Vision,
  Cambridge University Press.

Some First Computer Vision Software

- Open Computer Vision Library (OpenCV)
  - C++ library
  - has wrappers for Python & Octave
  - originally developed by Intel
  - v3.0 beta, 11/2014; `http://opencv.org`

Public data sets:

- ...

Summary (1/3)

- The **projective plane** $\mathbb{P}^2$ is an extension of the Euclidean plane with **ideal points**.
- Points and lines in $\mathbb{P}^2$ are parametrized by **homogenuous coordinates**.
- Each two parallels intersect in an ideal point, all ideal points form the **line at infinity** l_∞.
- Each circle contains two ideal points, the **circular points**, all lines through the circular points form the **dual conic to the circular points** C^*_∞.
- **Conics** are curves of order 2 (hyperbolas, parabolas, ellipsis), parametrized by a symmetric matrix C containing all points x with $x^T C x = 0$.

Summary (2/3)

- **Projectivities** H are invertibles mappings of $\mathbb{P}^2$ onto $\mathbb{P}^2$ that preserve lines.
- Lines a transform via $H^{-T}a$, conics C via $H^{-T}CH^{-1}$.
- There exist several subgroups of the group of projectivities:
  - **Isometries** **rotate** and **translate** figures.
    - preserving lengths
  - **Similarities** additionally **(isotropic) scale** figures.
    - preserving ratio of lengths, angle
  - **Affine transforms** additionally **non-isotropic scale** figures.
    - preserving ratio of lengths on parallel lines, parallel lines
  - **Projectivities** additionally **move the line at infinity**.
    - preserving cross ratio
- Any projectivity can be decomposed into a chain of an pure projectivie, a pure affine transform and a similarity.

Summary (3/3)

- Images distorted by an projective transformation can be **rectified** (i.e., undoes the projective transformation).
- **Affine rectification**
  - undoes a proper projective transformation
  - moves the **line at infinity** back to its **canonical position**.
  - allows to measure **affine properties**:
    - ratio of lengths on parallel lines, parallel lines
  - requires, e.g., **two pairs of parallel lines**.
- **Metric rectification**
  - undoes a proper affine transformation
  - moves the **dual conic to the circular points** back to its canonical position.
  - allows to measure **metric properties**:
    - angles, ratio of lengths
  - requires, e.g., two pairs of orthogonal lines.

Further Readings

- [HZ04, ch. 1 and 2].

References

Richard Hartley and Andrew Zisserman.
Multiple view geometry in computer vision.
Cambridge university press, 2004.