Outline

1. Points, Lines, Planes in Projective Space

2. Quadrics

2. Transformations

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2. Transformations

Objects in 2D Revisited

type	repr.	dim	dof	examples
points	$\mathbb{P}^2$	0	2	circular points I, J
lines	$\mathbb{P}^2$	1	2	line at inf. l_∞
point conics	$\mathrm{Sym}(\mathbb{P}^{2\times 2})$	1	5	
line conics	$\mathrm{Sym}(\mathbb{P}^{2\times 2})$	2	5	dual conic of circ. pts. C^*_∞

Note: The dimensionality applies to non-degenerate cases only.

Homogeneous Coordinates: Points

Inhomogeneous coordinates:

$$x \in \mathbb{R}^3$$

Homogeneous coordinates:

$$x \in \mathbb{P}^3 := \mathbb{R}^4 / \equiv$$

$$x \equiv y :\Longleftrightarrow \exists s \in \mathbb{R} \setminus \{0\} : sx = y, \quad x, y \in \mathbb{R}^4$$

Example:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ 8 \\ 12 \\ 16 \end{pmatrix} \text{ represent the same point in } \mathbb{P}^3$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} \text{ represent a different point in } \mathbb{P}^3$$

Dual of Points: Planes

Inhomogeneous coordinates:

$$p \in \mathbb{R}^4 : P_p := \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 = 0\}$$

Homogeneous coordinates:

$$p \in \mathbb{P}^3 : P_p := \{x \in \mathbb{P}^3 \mid p^T x = p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 = 0\}$$

- contains all finite points of $p' \in \kappa^{-1}(p)$: $P_{\kappa(p')} \supsetneqq \iota(P_{p'})$

Note: $\kappa : \mathbb{R}^4 \to \mathbb{P}^3, a \mapsto [a] := \{a' \in \mathbb{R}^4 \mid a' \equiv a\}$.

Intersecting Planes

Zeroset / **Null space**:

$$\text{Nul}(H) := \{p \in \mathbb{P}^3 \mid Hp = 0\}$$

All points incident to two planes p, q $(p \neq q)$:

$$PP(p,q) := \{x \in \mathbb{P}^3 \mid x \in P_p, x \in P_q\} = \{x \in \mathbb{P}^3 \mid p^T x = q^T x = 0\}$$

Can be represented as zeroset:

$$PP(p,q) = \text{Nul}(pq^T - qp^T)$$

- idea: represent lines as intersection of planes

All Planes Containing Two Points

All planes containing two points x, y $(x \neq y)$:

$$PP^*(x, y) := \{p \in \mathbb{P}^3 \mid x, y \in P_p\} = \{p \in \mathbb{P}^3 \mid p^T x = p^T y = 0\}$$

Can be represented as zeroset:

$$PP^*(x, y) = \text{Nul}(xy^T - yx^T)$$

- this is just the dual of "All points incident to two planes"
- idea: represent lines as intersection of planes
  - any two planes containing two points x, y will do

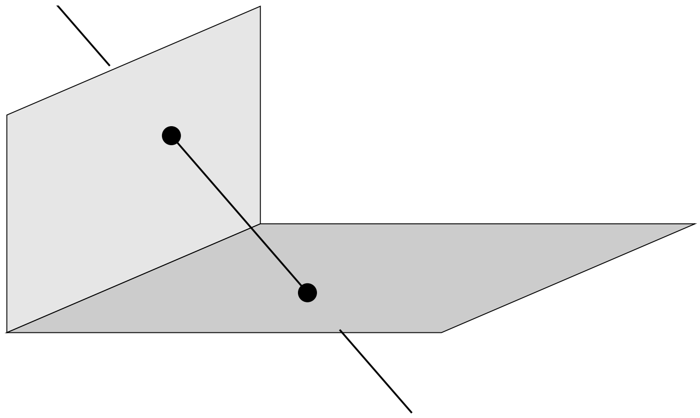
Plücker Matrix

For two points $x, y \in \mathbb{P}^3$:

$$\text{Plü}(x, y) := A := xy^T - yx^T$$

- skew symmetric: $A^T = -A$
  - esp. zero diagonal: $A_{i,i} = 0$.
- rank 2 (for $x \neq y$)

Lines have 4 Degrees of Freedom



[HZ04, p. 68]

Lines via Dual Plücker Matrices

Lines can be defined easily via spans:

$$\text{span}(x^1, x^2, \ldots, x^M) := \sum_{m=1}^{M} \mathbb{R}x^m := \{z \in \mathbb{R}^M \mid \exists s \in \mathbb{R}^M : z = \sum_{m=1}^{M} s_m x^m\}$$

$$l(x, y) := \text{span}(x, y)$$

Lines can be represented in 3D as zeroset of the **dual Plücker matrix**:

$$l(x, y) = \text{Nul}(\text{Plü}^*(x, y))$$

with

$$\text{Plü}^*(x, y) := A^* := \begin{pmatrix} 0 & A_{3,4} & A_{4,2} & A_{2,3} \\ -A_{3,4} & 0 & A_{1,4} & A_{3,1} \\ -A_{4,2} & -A_{1,4} & 0 & A_{1,2} \\ -A_{2,3} & -A_{3,1} & -A_{1,2} & 0 \end{pmatrix}$$

$$\text{and } \text{Plü}(x, y) := A := xy^T - yx^T \quad \text{(Plücker-Matrix)}$$

Lines via Dual Plücker Matrices

$$\begin{aligned} \mathrm{PP}(x,y) &= \mathrm{Nul}(A), \quad A = xy^T - yx^T \\ l(x,y) &= \mathrm{Nul}(A^*), \quad A^* = pq^T - qp^T, \quad p,q \in \mathrm{PP}(x,y) \end{aligned}$$

Now

$$\begin{aligned} A^*A =& (pq^T - qp^T)(xy^T - yx^T) \\ =& pq^T xy^T - pq^T yx^T - qp^T xy^T + qp^T yx^T = 0 \end{aligned}$$

therefore for all i, j, $i \neq j$:

$$0 = -(A^*A)_{i,j} = \sum_{k=1}^{4} A^*_{i,k} A_{j,k} = \sum_{k \notin \{i,j\}} A^*_{i,k} A_{j,k} \quad \text{as diagonals are zero}$$

i.e., $A^*_{i,k_1} A_{j,k_1} + A^*_{i,k_2} A_{j,k_2} = 0, \quad \{1,2,3,4\} = \{i,j,k_1,k_2\}$

and thus

$$\frac{A_{3,4}}{A^*_{1,2}} = \frac{A_{4,2}}{A^*_{1,3}} = \frac{A_{2,3}}{A^*_{1,4}} = \frac{A_{1,2}}{A^*_{3,4}} = \frac{A_{1,3}}{A^*_{4,2}} = \frac{A_{1,4}}{A^*_{2,3}}$$

Operations on Points, Lines & Planes

point x on plane p:	$p^T x = 0$
point x on line A^*:	$A^* x = 0$
plane p joining points x, y, z:	
plane p joining point x and line A^*:	$p = A^* x$
line A^* joining points x, y:	$A^* = \mathrm{Plü}^*(x,y)$
line A^* as intersection of planes p, q:	$A^* = pq^T - qp^T$
point x as intersection of plane p and line A:	$x = Ap$
line A is on plane p:	$Ap = 0$

Plane at Infinity p_∞

- All ideal points $(x_1, x_2, x_3, 0)^T$ form a plane, the **plane at infinity** $p_\infty := (0, 0, 0, 1)^T$.
-

$$\left.\begin{array}{l}\textbf{Two parallel planes}\\ \textbf{A parallel line and plane}\\ \textbf{Two parallel lines}\end{array}\right\}\text{intersect in}\left\{\begin{array}{l}\textbf{a line}\\ \textbf{a point}\\ \textbf{a point}\end{array}\right.\text{ on } p_\infty$$

- p_∞ is fixed under affine transformations.

Proofs: same as for the line at infinity in $\mathbb{P}^2$.

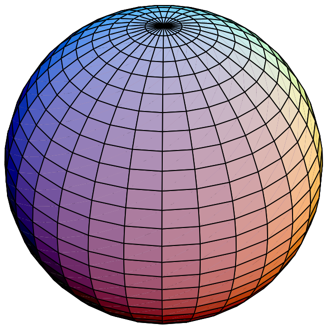
Outline

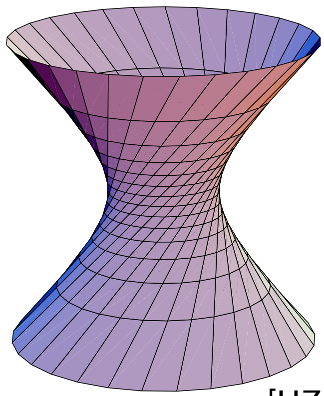
Quadrics

Quadratic surfaces:

$$\mathbf{Q}_Q := \{x \in \mathbb{P}^3 \mid x^T Q x = 0\}, \quad Q \in \mathrm{Sym}(\mathbb{P}^{4\times 4})$$

- 9 degrees of freedom
- 9 points in general position define a quadric
- The intersection of a plane p with a quadric Q is a conic
- A quadric Q transforms as $H^{-T}QH^{-1}$: $H(\mathbf{Q}_Q) = \mathbf{Q}_{H^{-T}QH^{-1}}$





[HZ04, p. 75]

Quadrics / Signature

$$\begin{aligned} Q &= USU^T && \text{SVD: } S \text{ diagonal, } UU^T = I \\ &= HS'H^T && S' \text{ diagonal with } S'_{i,i} \in \{+1, -1, 0\} \end{aligned}$$

signature of quadric Q:

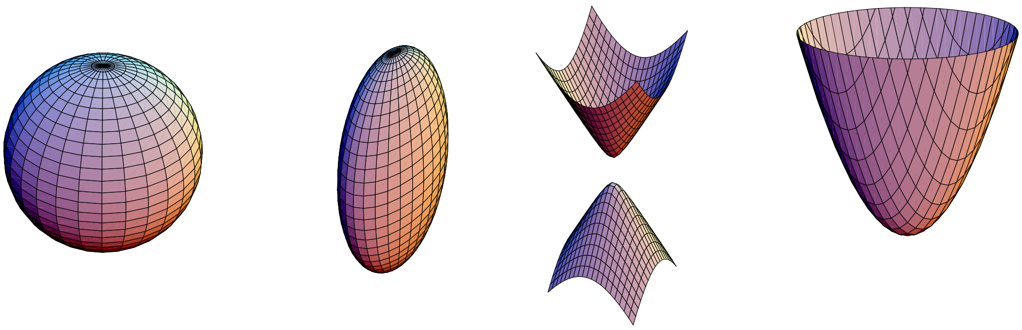
$$\sigma(Q) := |\{i \in \{1,2,3,4\} \mid S'_{i,i} = +1\}| - |\{i \in \{1,2,3,4\} \mid S'_{i,i} = -1\}|$$

Quadrics / Types

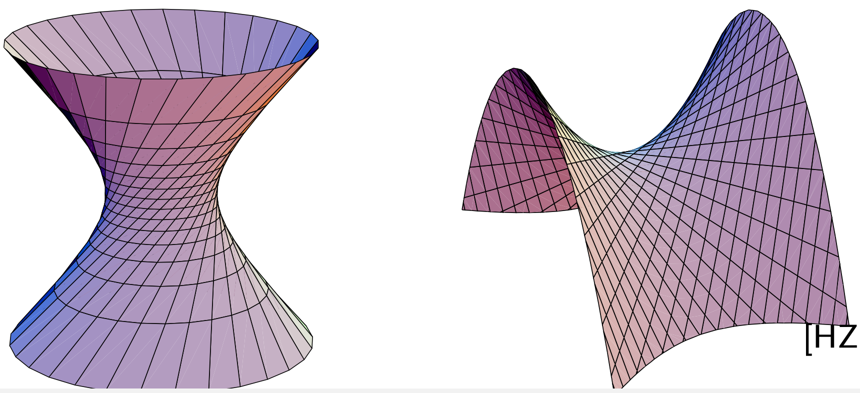
rank	σ	diagonal	equation	point set
4	4	(1, 1, 1, 1)	$x^2 + y^2 + z^2 + 1 = 0$	no real points
	2	(1, 1, 1, -1)	$x^2 + y^2 + z^2 - 1 = 0$	sphere
	0	(1, 1, -1, -1)	$x^2 + y^2 - z^2 - 1 = 0$	hyperboloid of one sheet
3	3	(1, 1, 1, 0)	$x^2 + y^2 + z^2 = 0$	one point $(0, 0, 0, 1)^T$
	1	(1, 1, -1, 0)	$x^2 + y^2 - z^2 = 0$	cone at origin
2	2	(1, 1, 0, 0)	$x^2 + y^2 = 0$	single line (z-axis)
	0	(1, -1, 0, 0)	$x^2 - y^2 = 0$	two planes $x = \pm y$
1	1	(1, 0, 0, 0)	$x^2 = 0$	one plane $x = 0$

Quadrics / Types

a) rank $= 4, \sigma = 2$: sphere / ellipsoid



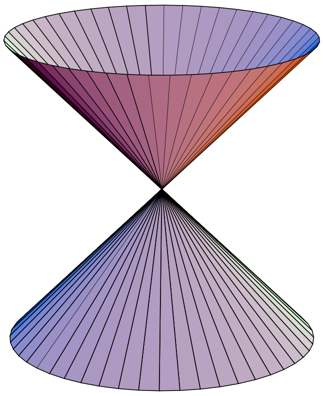
b) rank $= 4, \sigma = 0$: hyperboloid



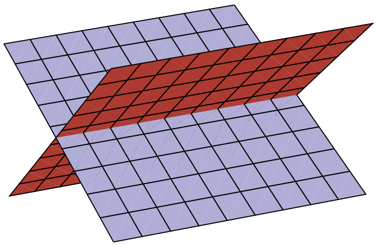
[HZ04, p. 75]

Quadrics / Types (2/2)

c) rank $= 3, \sigma = 1$: cone



d) rank $= 2, \sigma = 0$: two planes



[HZ04, p. 76]

Absolute Dual Quadric Q^*_∞

Plane/dual quadrics:

$$\mathbf{Q}^*_{Q*} := \{p \in \mathbb{P}^3 \mid p^T Q^* p = 0\}, \quad Q^* \in \mathrm{Sym}(\mathbb{P}^{4\times 4})$$

Absolute dual quadric:

$$Q^*_\infty := \begin{pmatrix} I & 0 \\ 0^T & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Absolute Dual Quadric Q^*_∞ Invariant under Similarity

The absolute dual quadric Q^*_∞ is invariant under projectivity H

$\Leftrightarrow$

H is a similarity.

proof:

$$H = \begin{pmatrix} A & t \\ v^T & v_4 \end{pmatrix},$$

$$HQ^*_\infty H^T = \begin{pmatrix} A & t \\ v^T & v_4 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0^T & 0 \end{pmatrix} \begin{pmatrix} A^T & v \\ t^T & v_4 \end{pmatrix} \begin{pmatrix} A^T & v \\ 0^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} AA^T & Av \\ v^T A^T & v^T v \end{pmatrix} \stackrel{!}{=} Q^*_\infty$$

$$\Leftrightarrow v = 0, AA^T = I, \text{i.e., } H \text{ is a similarity}$$

Absolute Dual Quadric Q^*_∞

- p_∞ is the nullvector of Q^*_∞.
- The **angle between two planes** is given by

$$\cos\theta(p, q) := \frac{p^T Q^*_\infty q}{\sqrt{p^T Q^*_\infty p \, q^T Q^*_\infty q}}$$

- esp. two planes p, q are orthogonal iff $p^T Q^*_\infty q = 0$.

proofs: as in $\mathbb{P}^2$.

Absolute Conic Ω_∞

$$\begin{aligned}\mathbf{C}_{\Omega_\infty} :=& \mathbf{Q}_{Q^*_\infty} \cap P_{p_\infty} \\ =& \{x \in \mathbb{P}^3 \mid x_1^2 + x_2^2 + x_3^2 = 0, x_4 = 0\}\end{aligned}$$

- H is a similarity transform $\Leftrightarrow$ Ω_∞ is invariant under H

Objects in 3D

type	repr.	dim	dof	examples
points	$\mathbb{P}^3$	0	3	
lines	$\mathrm{Skew}(\mathbb{P}^{4\times 4})$	1	4	
planes	$\mathbb{P}^3$	2	3	plane at inf. p_∞
point quadrics	$\mathrm{Sym}(\mathbb{P}^{4\times 4})$	2	9	
plane quadrics	$\mathrm{Sym}(\mathbb{P}^{4\times 4})$	3	9	absolute dual quadric Q^*_∞
conic	$p \cap Q$	1	8	absolute conic Ω_∞

Note: The dimensionality applies to non-degenerate cases only.

Outline

1. Points, Lines, Planes in Projective Space

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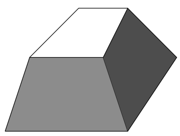
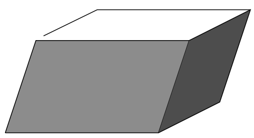
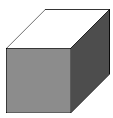
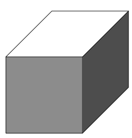
Hierarchy of Transformations

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

[HZ04, p. 78]

Rotations in 3D

Rotations in 3D can be described by a **rotation axis** and a **rotation angle**.

Pure rotations (rotations along an axis through the origin) can be described by

1. a **rotation axis direction** (an axis through the origin) and a **rotation angle**, or
2. **Euler-Tait-Bryan angles**:

$$R = R_z(\gamma)R_y(\beta)R_x(\alpha),$$

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R_y(\beta) := \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}, \quad R_z(\gamma) := \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. a **proper orthogonal matrix**:

$$R \in \mathbb{R}^{3\times 3} : RR^T = R^T R = I, \det R = 1$$

Pure rotations have 3 dof.

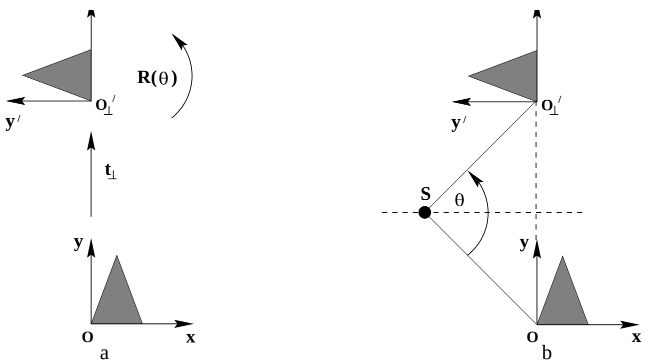
The Screw Decomposition

Any Euclidean transformation, i.e., a 3D rotation R followed by a translation t, can be represented as

- a rotation followed by
- a translation along the same axis (called **skrew axis**)

Proof:

1. if t is orthogonal to the rotation axis of R: planar transformation.



2. generally: decompose t into $t_{\text{orthogonal}}$ and t_{parallel}. [HZ04, p. 79]

Summary (1/2)

- The **projective space** $\mathbb{P}^3$ is an extension of the Euclidean space $\mathbb{R}^3$ with **ideal points**.
- Points and planes in $\mathbb{P}^2$ are parametrized by **homogenuous coordinates**.
- Each two parallel lines intersect in an ideal point, each two parallel planes intersect in a line of ideal points, all ideal points form the **plane at infinity** p_∞.
- **Quadrics** are surfaces of order 2 (hyperboloid, paraboloid, ellipsoid), parametrized by a symmetric matrix Q containing all points x with $x^T Qx = 0$.

Summary (2/2)

- **Projectivities** H are invertibles mappings of $\mathbb{P}^3$ onto $\mathbb{P}^3$ that preserve lines.
- Lines a transform via $H^{-T}a$, quadrics Q via $H^{-T}QH^{-1}$.
- There exist several subgroups of the group of projectivities:
  - **Isometries** **rotate** and **translate** figures.
    - preserving lengths
  - **Similarities** additionally **(isotropic) scale** figures.
    - preserving ratio of lengths, angle, the plane at infinity p_∞
  - **Affine transforms** additionally **non-isotropic scale** figures.
    - preserving ratio of lengths on parallel lines, parallel lines, the absolute conic Ω_∞
  - **Projectivities** additionally **move the plane at infinity**.
    - preserving cross ratio
- Any projectivity can be decomposed into a chain of an pure projectivie, a pure affine transform and a similarity.

Further Readings

- [HZ04, ch. 3].
- for the derivation of the dual Plücker coordinates [Cox98, p. 88f]

References

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