Outline

Objects to estimate from data

- a 2D projectivity
- a 3D to 2D projection (camera)
- the Fundamental Matrix
- the Trifocal Tensor

Data:

- N pairs x_n,x'_n of corresponding points in two images ($n = 1, \ldots, N$)

Note: The Trifocal Tensor represents a relation between three images and thus requires N **triples** of corresponding points x_n,x'_n, x''_n in **three images** ($n = 1, \ldots, N$).

Outline

From Corresponding Points to Linear Equations (1/2)

Inhomogeneous coordinates:

$$x'_n \overset{!}{=} \hat{x}'_n := H x_n, \quad n = 1, \ldots, N$$

$$= \begin{pmatrix} x_n^T & 0^T & 0^T \\ 0^T & x_n^T & 0^T \\ 0^T & 0^T & x_n^T \end{pmatrix} h, \quad h := \text{vect}(H) := \begin{pmatrix} H_{1,1} \\ H_{1,2} \\ H_{1,3} \\ H_{2,1} \\ \vdots \\ H_{3,3} \end{pmatrix}$$

Homogeneous coordinates:

$$x'_{n,i} : x'_{n,j} = \hat{x}'_{n,i} : \hat{x}'_{n,j}, \quad \forall i,j \in \{1,2,3\}, i \neq j$$

$$x'_{n,i}\hat{x}'_{n,j} - x'_{n,j}\hat{x}'_{n,i} = 0, \quad \text{and one equation is linear dependent}$$

$$\rightsquigarrow 0 \overset{!}{=} \underbrace{\begin{pmatrix} 0^T & -x'_{n,3} x_n^T & x'_{n,2} x_n^T \\ x'_{n,3} x_n^T & 0^T & -x'_{n,1} x_n^T \end{pmatrix}}_{=:A(x_n, x'_n)} h$$

From Corresponding Points to Linear Equations (2/2)

$$A(x_n, x'_n)h \overset{!}{=} 0, \quad n = 1, \ldots, N$$

$$\underbrace{\begin{pmatrix} A(x_1, x'_1) \\ A(x_2, x'_2) \\ \vdots \\ A(x_N, x'_N) \end{pmatrix}}_{=:A(x_{1:N}, x'_{1:N})} h = 0$$

- to estimate a general projectivity we need 4 points (8 equations, 8 dof)
- we are looking for non-trivial solutions $h \neq 0$.

More than 4 Points & Noise: Overdetermined

- For $N > 4$ points and **exact coordinates**, the system $Ah = 0$ still has rank 8 and a non-trivial solution $h \neq 0$.
- But for $N > 4$ points and **noisy coordinates**, the system $Ah = 0$ is overdetermined and (in general) has only the trivial solution $h = 0$.

Relax the objective $Ah = 0$ to

$$\begin{aligned} \arg\min_{h:||h||=1} ||Ah|| &= \arg\min_{h} \frac{||Ah||}{||h||} \\ &= \text{(normed) eigenvector to smallest eigenvalue} \end{aligned}$$

and solve via SVD:

$$\begin{aligned} A^T A &= USU^T, \quad S = \text{diag}(s_1, \ldots, s_9), s_i \geq s_{i+1} \forall i, UU^T = I \\ h &:= U_{9,1:9} \end{aligned}$$

Degenerate Configurations: Underdetermined

- If three of the four points are collinear (in both images), A will have rank < 8 and thus h underdetermined, and thus there is no unique solution for h.

Degenerate Configuration:
Corresponding points that do not uniquely determine a transformation (in a particular class of transformations).

Direct Linear Transformation Algorithm (DLT)

1: **procedure** EST-2D-PROJECTIVITY-DLT$(x_1, x'_1, x_2, x'_2, \ldots, x_N, x'_N \in \mathbb{P}^2)$

2: $$A := \begin{pmatrix} A(x_1, x'_1) \\ A(x_2, x'_2) \\ \vdots \\ A(x_N, x'_N) \end{pmatrix} = \begin{pmatrix} 0^T & -x'_{1,3} x_1^T & x'_{1,2} x_1^T \\ x'_{1,3} x_1^T & 0^T & -x'_{1,1} x_1^T \\ 0^T & -x'_{2,3} x_2^T & x'_{2,2} x_2^T \\ x'_{2,3} x_2^T & 0^T & -x'_{2,1} x_2^T \\ \vdots & & \\ 0^T & -x'_{N,3} x_N^T & x'_{N,2} x_N^T \\ x'_{N,3} x_N^T & 0^T & -x'_{N,1} x_N^T \end{pmatrix}$$

3: $(U, S) := \text{SVD}(A^T A)$

4: $h := U_{9,1:9}$

5: **return** $H := \begin{pmatrix} h_{1:3}^T \\ h_{4:6}^T \\ h_{7:9}^T \end{pmatrix}$

Note: Do not use this unnormalized version of DLT, but the one in section 3.

Outline

Algebraic Distance

- the loss minimized by DLT, represented as distance between
  - x': point in 2nd image
  - $\hat{x}' := Hx$: estimated position of x' by H

$$\begin{aligned}\ell_{\text{alg}}(H; x, x') &:= \|A(x', x)h\|^2 \\ &= \left\| \begin{pmatrix} 0^T & -x_3' x^T & x_2' x^T \\ x_3' x^T & 0^T & -x_1' x^T \end{pmatrix} h \right\|^2 \\ &= \left\| \begin{pmatrix} -x_3' \hat{x}_2' + x_2' \hat{x}_3' \\ x_3' \hat{x}_1' - x_1' \hat{x}_3' \end{pmatrix} \right\|^2 \\ &= d_{\text{alg}}(x', \hat{x}')^2\end{aligned}$$

with

$$d_{\text{alg}}(x, y) := \sqrt{a_1^2 + a_2^2}, \quad (a_1, a_2, a_3)^T = x \times y$$

Geometric Distances: Transfer Errors

Transfer Error in One Image (2nd image):

$$\ell_{\text{trans1}}(H; x, x') := d(x', Hx)^2 = d(x', \hat{x}')^2$$

with Euclidean distance in inhomogeneous coordinates

$$d(x, y) := \sqrt{(x_1/x_3 - y_1/y_3)^2 + (x_2/x_3 - y_2/y_3)^2}$$
$$= \frac{1}{|x_3||y_3|} d_{\text{alg}}(x, y)$$

- DLT/algebraic error equals geometric error for affine transformations ($x_3 = y_3 = 1$)

Symmetric Transfer Error:

$$\ell_{\text{strans}}(H; x, x') := d(x, H^{-1}x')^2 + d(x', Hx)^2$$
$$= d(x, \hat{x})^2 + d(x', \hat{x}')^2, \quad \hat{x} := H^{-1}x'$$

Transfer Errors: Probabilistic Interpretation

Assume

- measurements x_n in the 1st image are noise-free,
- measurements x'_n in the 2nd image are distributed Gaussian around true values Hx_n:

$$p(x'_n \mid Hx_n, \sigma^2) = \frac{1}{2\pi\sigma^2} e^{-d(x'_n, Hx_n)^2/(2\sigma^2)}$$

log-likelihood for Transfer Error in One Image:

$$p(H \mid x_{1:N}, x'_{1:N}) = \frac{p(x_{1:N}, x'_{1:N} \mid H) p(H)}{p(x_{1:N}, x'_{1:N})} \qquad \text{Bayes}$$
$$\propto p(x_{1:N}, x'_{1:N} \mid H) p(H) \propto \quad p(x'_{1:N} \mid H, x_{1:N}) p(H)$$
$$= p(H) \prod_{n=1}^N p(x'_n \mid H, x_n) \propto \quad \prod_{n=1}^N p(x'_n \mid H, x_n)$$

$$\log p(H \mid x_{1:N}, x'_{1:N}) \propto - \sum_{n=1}^N d(x'_n, Hx_n)^2 \qquad = \text{transfer error}$$

Reprojection Error

- additionally to projectivity H, also find noise-free / perfectly matching pairs $\hat{x}, \hat{x}'$:

$$\text{minimize } \ell_{rep}(H, \hat{x}_1, \hat{x}'_1, \ldots, \hat{x}_N, \hat{x}'_N) := \sum_{n=1}^{N} d(x_n, \hat{x}_n)^2 + d(x'_n, \hat{x}'_n)^2$$

w.r.t.

$$\hat{x}'_n = H\hat{x}_n, \quad n = 1, \ldots, N$$

over

$$H, \hat{x}_1, \hat{x}'_1, \ldots, \hat{x}_N, \hat{x}'_N$$

Reprojection Error:

$$\ell_{\text{rep}}(H, \hat{x}, \hat{x}'; x, x') := d(x, \hat{x})^2 + d(x', \hat{x}')^2, \quad \text{with } \hat{x}' = H\hat{x}$$

- analogue probabilistic interpretation:
  - measurements x, x' are Gaussian around true values $\hat{x}, \hat{x}'$

Outline

Are Solutions Invariant under Transformations?

- Given corresponding points x_n, x'_n,
  a method such as DLT will find a projectivity H.
- Now assume
  - the first image is transformed by projectivity T,
  - the second image is transformed by projectivity T'

  before we apply the estimation method.
  - Corresponding points now will be $\tilde{x}_n := Tx_n, \tilde{x}'_n := T'x'_n$
- Let $\tilde{H}$ be the projectivity estimated by the method applied to $\tilde{x}_n, \tilde{x}'_n$.
- Is it guaranteed that H and $\tilde{H}$ are "the same" (equivalent) ?

$$\tilde{H} \stackrel{?}{=} T'HT^{-1}$$

- This may depend on the class of projectivities allowed for T, T'.
  - at least invariance under similarities would be useful !

DLT is not Invariant under Similarities

- If T' is a similarity transformation with scale factor s
  and T any projectivity, then one can show

$$||\tilde{A}\tilde{h}|| = s||Ah||$$

- But solutions H and $\tilde{H}$ will not be equivalent nevertheless,
  as DLT minimizes under constraint $||h|| = 1$
  and this constraint is not scaled with s !
- So DLT is not invariant under similarity transforms.

Note: $\tilde{A} := A(\tilde{x}_., \tilde{x}'_.), \tilde{h} := \text{vect}(\tilde{H})$

Transfer/Reprojection Errors are Invariant under Similarities

- If T' is Euclidean:

$$d(\tilde{x}'_n, \tilde{H}\tilde{x}_n)^2 = d(T'x'_n, T'HT^{-1}Tx_n)^2$$
$$= {x'_n}^T T'^T T'HT^{-1}Tx_n = x'_n Hx_n = d(x'_n, Hx_n)^2$$

- If T' is a similarity with scale factor s:

$$d(\tilde{x}'_n, \tilde{H}\tilde{x}_n)^2 = d(T'x'_n, T'HT^{-1}Tx_n)^2$$
$$= x'_n T'^T T'HT^{-1}Tx_n = x'_n s^2 Hx_n = s^2 d(x'_n, Hx_n)^2$$

- Error is just scaled, so attains minimum at same position.
  $\rightsquigarrow$ Transfer/Reprojection Errors are invariant under similarities.

DLT with Normalization

- Image coordinates of corresponding points are usually finite:
  $x = (x_1, x_2, 1)^T$,
  thus have different scale $(100, 100, 1)$ when measured in pixels.
- Therefore, entries in $A(x, x')$ will have largely different scale:

$$A(x, x') = \begin{pmatrix} 0^T & -x'_3 x^T & x'_2 x^T \\ x'_3 x^T & 0^T & -x'_1 x^T \end{pmatrix} = \begin{pmatrix} 0^T & -x^T & x'_2 x^T \\ x^T & 0^T & -x'_1 x^T \end{pmatrix}$$

  - some in 100s (x^T), some in 10.000s ($x'_2 x^T, -x'_1 x^T$)

DLT with Normalization

- normalize $x_{1:N}$:

$$\tilde{x}_{1:N} := \text{normalize}(x_{1:N}) := \left(\frac{x_n - \mu(x_{1:N})}{\tau(x_{1:N})/\sqrt{2}}\right)_{n=1,\ldots,N},$$

  with

$$\mu(x_{1:N}) := \frac{1}{N}\sum_{n=1}^{N} x_n \qquad \text{centroid/mean}$$

$$\tau(x_{1:N}) := \frac{1}{N}\sum_{n=1}^{N} d(x_n, \mu(x_{1:N})) \qquad \text{avg. distance to centroid}$$

- afterwards:

$$\mu(\tilde{x}_{1:N}) = 0, \quad \tau(\tilde{x}_{1:N}) = \sqrt{2}$$

- Normalization is a similarity transform:

$$T := T_{\text{norm}}(x_{1:N}) := \begin{pmatrix} \sqrt{2}/\tau(x_{1:N})I & -\mu(x_{1:N})\sqrt{2}/\tau(x_{1:N}) \\ 0 & 1 \end{pmatrix}$$

DLT with Normalization / Algorithm

1: **procedure** EST-2D-PROJECTIVITY-DLTN$(x_1, x'_1, x_2, x'_2, \ldots, x_N, x'_N \in \mathbb{P}^2)$

2: $\quad T := T_{\text{norm}}(x_{1:N}) := \begin{pmatrix} \sqrt{2}/\tau(x_{1:N})I & -\mu(x_{1:N})\sqrt{2}/\tau(x_{1:N}) \\ 0 & 1 \end{pmatrix}$

3: $\quad T' := T_{\text{norm}}(x'_{1:N}) := \begin{pmatrix} \sqrt{2}/\tau(x'_{1:N})I & -\mu(x'_{1:N})\sqrt{2}/\tau(x'_{1:N}) \\ 0 & 1 \end{pmatrix}$

4: $\quad \tilde{x}_n := T x_n \quad \forall n = 1, \ldots, N$ ▷ normalize x_n

5: $\quad \tilde{x}'_n := T' x'_n \quad \forall n = 1, \ldots, N$ ▷ normalize x'_n

6: $\quad \tilde{H} :=$ est-2d-projectivity-dlt$(\tilde{x}_1, \tilde{x}'_1, \tilde{x}_2, \tilde{x}'_2, \ldots, \tilde{x}_N, \tilde{x}'_N)$

7: $\quad H := T'^{-1}\tilde{H}T$ ▷ unnormalize $\tilde{H}$

8: $\quad$ **return** H

Outline

Types of Problems

- The transformation estimation problem for the
  - **algebraic distance/loss** can be cast into a single
  - **linear system of equations (DLTn)**.
- The transformation estimation problem for the
  - **transfer distance/loss** as well as for the
  - **reconstruction loss** is more complicated and has to be handled by an explicit
  - **iterative minimization procedure**.

Minimization **Objectives** $f : \mathbb{R}^M \to \mathbb{R}$

a) transfer distance in one image:

$$\text{minimize } f(H) := \sum_{n=1}^{N} d(x'_n, Hx_n)^2$$

b) symmetric transfer distance:

$$\text{minimize } f(H) := \sum_{n=1}^{N} d(x'_n, Hx_n)^2 + d(x_n, H^{-1}x'_n)^2$$

c) reconstruction loss:

$$\text{minimize } f(H, \hat{x}_{1:N}) := \sum_{n=1}^{N} d(x_n, \hat{x}_n)^2 + d(x'_n, H\hat{x}_n)^2$$

- x_n, x'_n are constants, $H, \hat{x}_{1:N}$ variables
- a), b) have $M := 9$ **parameters** / variables
  - as H as only 8 dof, the objective is slightly **overparametrized**
- c) has $M := 2N + 9$ **parameters** / variables
  - allowing only finite points for $\hat{x}_n$

Objectives of type $f = e^T e$ (1/3)

All three objectives f are L_2 norms of (parametrized) vectors, i.e. can be written as

$$f(x) = e(x)^T e(x), \quad h : \mathbb{R}^M \to \mathbb{R}^N$$

a) transfer distance in one image:

$$\begin{aligned}\text{minimize } f(H) &:= \sum_{n=1}^{N} d(x'_n, Hx_n)^2 \\ &= e(H)^T e(H),\end{aligned}$$

$$e(H) := \begin{pmatrix} x'_{1,1}/x'_{1,3} - (Hx_1)_1/(Hx_1)_3 \\ x'_{1,2}/x'_{1,3} - (Hx_1)_2/(Hx_1)_3 \\ \vdots \\ x'_{N,1}/x'_{N,3} - (Hx_N)_1/(Hx_N)_3 \\ x'_{N,2}/x'_{N,3} - (Hx_N)_2/(Hx_N)_3 \end{pmatrix}$$

Objectives of type $f = e^T e$ (2/3)

b) symmetric transfer distance:

$$\text{minimize } f(H) := \sum_{n=1}^{N} d(x'_n, Hx_n)^2 + d(x_n, H^{-1}x'_n)^2 = e(H)^T e(H),$$

$$e(H) := \begin{pmatrix} x'_{1,1}/x'_{1,3} - (Hx_1)_1/(Hx_1)_3 \\ x'_{1,2}/x'_{1,3} - (Hx_1)_2/(Hx_1)_3 \\ \vdots \\ x'_{N,1}/x'_{N,3} - (Hx_N)_1/(Hx_N)_3 \\ x'_{N,2}/x'_{N,3} - (Hx_N)_2/(Hx_N)_3 \\ x_{1,1}/x_{1,3} - (H^{-1}x'_1)_1/(H^{-1}x'_1)_3 \\ x_{1,2}/x_{1,3} - (H^{-1}x'_1)_2/(H^{-1}x'_1)_3 \\ \vdots \\ x_{N,1}/x_{N,3} - (H^{-1}x'_N)_1/(H^{-1}x'_N)_3 \\ x_{N,2}/x_{N,3} - (H^{-1}x'_N)_2/(H^{-1}x'_N)_3 \end{pmatrix}$$

Objectives of type $f = e^T e$ (3/3)

c) reconstruction loss:

$$\text{minimize } f(H, \hat{x}_{1:N}) := \sum_{n=1}^{N} d(x_n, \hat{x}_n)^2 + d(x'_n, H\hat{x}_n)^2 = e(H)^T e(H),$$

$$e(H) := \begin{pmatrix} x'_{1,1}/x'_{1,3} - (H\hat{x}_1)_1/(H\hat{x}_1)_3 \\ x'_{1,2}/x'_{1,3} - (H\hat{x}_1)_2/(H\hat{x}_1)_3 \\ \vdots \\ x'_{N,1}/x'_{N,3} - (H\hat{x}_N)_1/(H\hat{x}_N)_3 \\ x'_{N,2}/x'_{N,3} - (H\hat{x}_N)_2/(H\hat{x}_N)_3 \\ x_{1,1}/x_{1,3} - \hat{x}_{1,1} \\ x_{1,2}/x_{1,3} - \hat{x}_{1,2} \\ \vdots \\ x_{N,1}/x_{N,3} - \hat{x}_{N,1} \\ x_{N,2}/x_{N,3} - \hat{x}_{N,2} \end{pmatrix}$$

Minimizing f (I): Gradient Descent

To minimize $f : \mathbb{R}^M \rightarrow \mathbb{R}$ over $x \in \mathbb{R}^M$ **Gradient Descent**

1. starts at a random **starting point** $x_0 \in \mathbb{R}^M$

$$t := 0, \quad x^{(t)} := x_0$$

2. computes as **descent direction** $d^{(t)}$ **at** $x^{(t)}$
   — direction where f decreases —
   the **gradient of** f:

$$d^{(t)} := -g^{(t)} := -\nabla_x f|_{x^{(t)}} := -(\frac{\partial f}{\partial x_m}(x^{(t)}))_{m=1,\ldots,M}$$

3. moves into the descent direction:

$$x^{(t+1)} := x^{(t)} + d$$

Beware:

- f decreases only in the neighborhood of $x^{(t)}$
- A full gradient step may be too large and **not** leading to a decrease !

Minimizing f (I): Gradient Descent w. Steplength Control

To minimize $f : \mathbb{R}^M \rightarrow \mathbb{R}$ over $x \in \mathbb{R}^M$ **Gradient Descent**

1. starts at a random **starting point** $x_0 \in \mathbb{R}^M$

$$t := 0, \quad x^{(t)} := x_0$$

2. computes as **descent direction** $d^{(t)}$ **at** $x^{(t)}$
   — direction where f decreases —
   the **gradient of** f:

$$d^{(t)} := -g^{(t)} := -\nabla_x f|_{x^{(t)}} := -(\frac{\partial f}{\partial x_m}(x^{(t)}))_{m=1,\ldots,M}$$

3. finds a steplength $\alpha \in \mathbb{R}^+$ so that f actually decreases:

$$\alpha := \max\{\alpha := 2^{-k} \mid k = 0, 1, 2, \ldots, f(x + \alpha d) < f(x)\}$$

4. moves a step into the descent direction:

$$x^{(t+1)} := x^{(t)} + \alpha d$$

Minimizing f (I): Gradient Descent / Algorithm

procedure MIN-GD($f : \mathbb{R}^M \rightarrow \mathbb{R}, x_0 \in \mathbb{R}^M, \nabla_x f : \mathbb{R}^M \rightarrow \mathbb{R}^M, \epsilon \in \mathbb{R}^+$)
    $x := x_0$
    do
        $d := -\nabla_x f|_x$
        $\alpha := 1$
        while $f(x + \alpha d) \geq f(x)$ **do**
            $\alpha := \alpha/2$
        $x := x + \alpha d$
    while $||d|| > \epsilon$
    return x

Minimizing f (II): Newton

The Newton algorithm computes a better descent direction:

- approximate f by the **quadratic Taylor expansion at $x^{(t)}$**:

$$\begin{aligned} f(x+d) \approx \tilde{f}(d) &:= f(x^{(t)}) + \nabla_x f|_{x^{(t)}}^T d + \frac{1}{2} d^T \nabla_x^2 f|_{x^{(t)}}^T d \\ &= f(x^{(t)}) + g_{x^{(t)}}^T d + \frac{1}{2} d^T H_{x^{(t)}} d \end{aligned}$$

  where

$$\nabla_x^2 f|_x := H_x := \left(\frac{\partial^2 f}{\partial x_m \partial x_k}\right)_{m,k=1,\ldots,M} \quad \textbf{Hessian of } f$$

- the approximation attains its minimum at

$$\begin{aligned} 0 &\stackrel{!}{=} \nabla_d \tilde{f}(d) = g_{x^{(t)}} + H_{x^{(t)}} d \\ H_{x^{(t)}} d &= - g_{x^{(t)}} \qquad \textbf{normal equations} \end{aligned}$$

- solve this linear system of equations to find descent direction

Minimizing f (II): Newton / Algorithm

procedure $\textsc{Min-Newton}(f : \mathbb{R}^M \rightarrow \mathbb{R}, x_0 \in \mathbb{R}^M,$
$\quad \nabla_x f : \mathbb{R}^M \rightarrow \mathbb{R}^M, \nabla_x^2 f : \mathbb{R}^M \rightarrow \mathbb{R}^{M \times M}, \epsilon \in \mathbb{R}^+)$
$\quad x := x_0$
$\quad$ **do**
$\quad\quad g := \nabla_x f|_x$
$\quad\quad H := \nabla_x^2 f|_x$
$\quad\quad d := \text{solve}_d(Hd = -g)$
$\quad\quad \alpha := 1$
$\quad\quad$ **while** $f(x + \alpha d) \geq f(x)$ **do**
$\quad\quad\quad \alpha := \alpha / 2$
$\quad\quad x := x + \alpha d$
$\quad$ **while** $||d|| > \epsilon$
$\quad$ **return** x

Minimizing $f = e^T e$ (I): Gauss-Newton

Gauss-Newton is

- a **specialization of the Newton algorithm**
- for **objectives of type** $f(x) = e(x)^T e(x)$
- that **approximates the Hessian**:

$$\nabla_x f|_x = 2 \nabla_x e|_x^T e(x)$$
$$\nabla_x^2 f|_x = 2 \nabla_x e|_x^T \nabla_x e|_x + 2 \nabla_x^2 e|_x^T e(x)$$

Now approximate e by a linear Taylor expansion, i.e.

$$\nabla_x^2 e|_x \approx 0$$
$$\rightsquigarrow \quad \nabla_x^2 f|_x \approx 2 \nabla_x e|_x^T \nabla_x e|_x$$

- all we need is the gradient of e !

Minimizing $f = e^T e$ (I): Gauss-Newton / Algorithm

```
procedure MIN-GAUSS-NEWTON(e : ℝ^M → ℝ^N, x_0 ∈ ℝ^M, ∇_x e : ℝ^M → ℝ^{N×M}, ϵ ∈ ℝ^+)
    x := x_0
    do
        J := ∇_x e|_x
        g := J^T e(x)
        H := J^T J
        d := solve_d(Hd = −g)
        α := 1
        while e(x + αd)^T e(x + αd) ≥ e(x)^T e(x) do
            α := α/2
        x := x + αd
    while ||d|| > ϵ
    return x
```


Minimizing $f = e^T e$ (II): Levenberg-Marquardt

- slight variation of the Gauss-Newton method

$$J^T J\, d = -g \qquad \text{Gauss-Newton Normal Eq.}$$

$$(J^T J + \lambda I)\, d = -g \qquad \text{Levenberg-Marquardt Normal Eq.}$$

- if new objective value is worse, try again with larger λ
  - for large λ: equivalent to Gradient descent with small stepsize $1/\lambda$

$$(J^T J + \lambda I) \approx \lambda I, \qquad (J^T J + \lambda I)\, d = -g \qquad \leadsto d = -\frac{1}{\lambda} g$$

- once new objective value is smaller, accept and decrease λ
  - for small λ: equivalent to Gauss-Newton with (large) stepsize 1

Minimizing $f = e^T e$ (I): Levenberg-Marquardt / Algorithm

```
procedure MIN-LEVENBERG-MARQUARDT(e : ℝ^M → ℝ^N, x_0 ∈ ℝ^M, ∇_x e : ℝ^M → ℝ^{N×M}, ε ∈ ℝ^+)
    x := x_0
    λ := 1
    do
        J := ∇_x e|_x
        g := J^T e(x)
        λ := (λ/10)/10
        do
            H := J^T J + λI
            d := solve_d(Hd = −g)
            λ := 10λ
        while e(x + d)^T e(x + d) ≥ e(x)^T e(x)
        x := x + d
    while ||d|| > ε
    return x
```


Example: Reconstruction Loss (1/2)

$$e(H) := \begin{pmatrix} x'_{1,1}/x'_{1,3} - (H\hat{x}_1)_1/(H\hat{x}_1)_3 \\ x'_{1,2}/x'_{1,3} - (H\hat{x}_1)_2/(H\hat{x}_1)_3 \\ \vdots \\ x'_{N,1}/x'_{N,3} - (H\hat{x}_N)_1/(H\hat{x}_N)_3 \\ x'_{N,2}/x'_{N,3} - (H\hat{x}_N)_2/(H\hat{x}_N)_3 \\ x_{1,1}/x_{1,3} - \hat{x}_{1,1} \\ x_{1,2}/x_{1,3} - \hat{x}_{1,2} \\ \vdots \\ x_{N,1}/x_{N,3} - \hat{x}_{N,1} \\ x_{N,2}/x_{N,3} - \hat{x}_{N,2} \end{pmatrix} = \text{vect}(\begin{pmatrix} e^1_{1:N,1:2} \\ e^2_{1:N,1:2} \end{pmatrix})$$

with

$$e^1_{n,i} := x'_{n,i}/x'_{n,3} - (H\hat{x}_n)_i/(H\hat{x}_n)_3$$
$$e^2_{n,i} := x_{n,i}/x_{n,3} - \hat{x}_{n,i}$$

Example: Reconstruction Loss (2/2)

$$e^1_{n,i} := \frac{x'_{n,i}}{x'_{n,3}} - \frac{(H\hat{x}_n)_i}{(H\hat{x}_n)_3}$$

$$e^2_{n,i} := x_{n,i}/x_{n,3} - \hat{x}_{n,i}$$

$$\nabla_{\hat{x}_{\tilde{n},\tilde{i}}} e^1_{n,i} = \begin{cases} -\frac{H_{i,\tilde{i}}}{(H\hat{x}_n)_3} + \frac{(H\hat{x}_n)_i}{(H\hat{x}_n)_3^2} H_{3,\tilde{i}}, & \text{if } \tilde{n} = n \\ 0, & \text{else} \end{cases}$$

$$\nabla_{\hat{x}_{\tilde{n},\tilde{i}}} e^2_{n,i} = \begin{cases} -1, & \text{if } \tilde{n} = n, \tilde{i} = i \\ 0, & \text{else} \end{cases}$$

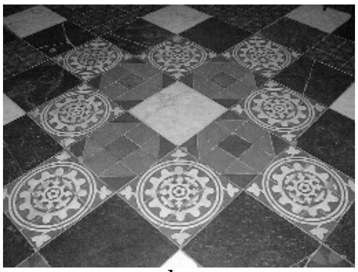
$$\nabla_{H_{\tilde{i},\tilde{j}}} e^1_{n,i} = -\delta(\tilde{i} = i)\frac{\hat{x}_{n,\tilde{j}}}{(H\hat{x}_n)_3} + \delta(\tilde{i} = 3)\frac{(H\hat{x}_n)_i}{(H\hat{x}_n)_3^2}\hat{x}_{n,3}$$

$$\nabla_{H_{\tilde{i},\tilde{j}}} e^2_{n,i} = 0$$

Note: $(H\hat{x}_n)_i = \sum_{j=1}^3 H_{i,j}\hat{x}_{n,j}$.

Example: Comparison of Different Methods


a

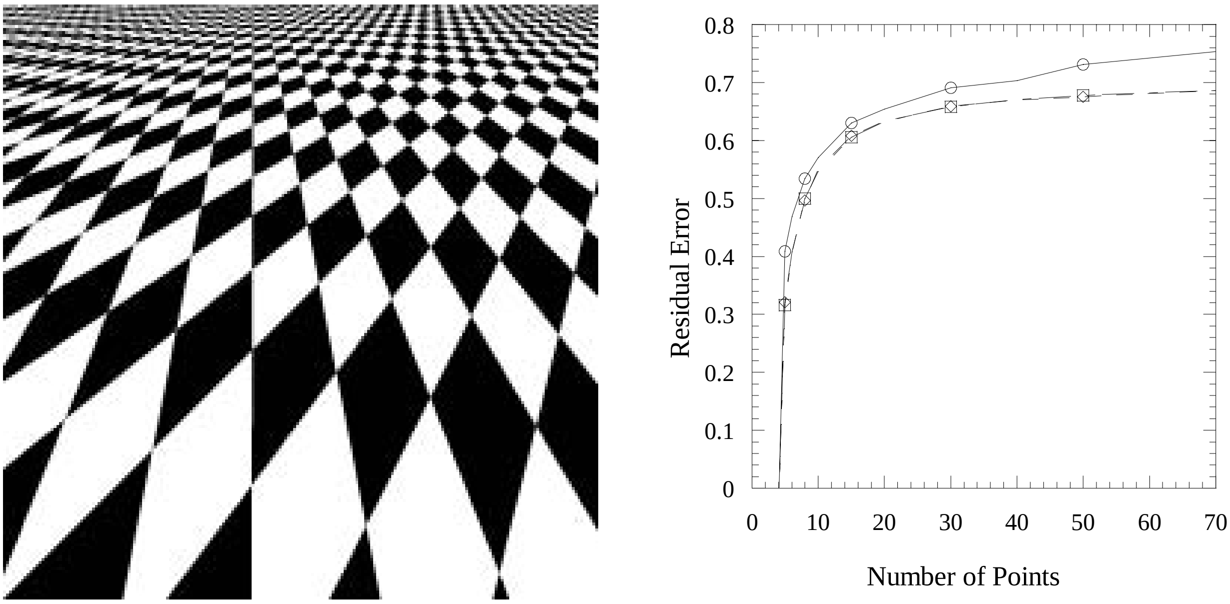

b


c

method	residual error in pixels pair a,b	pair a,c
DLT unnormalized	0.4080	26.2056
DLT normalized	0.4078	0.6602
Transfer distance in one image	0.4077	0.6602
Reconstruction loss	0.4078	0.6602
affine	6.0095	2.8481

[HZ04, p. 115]

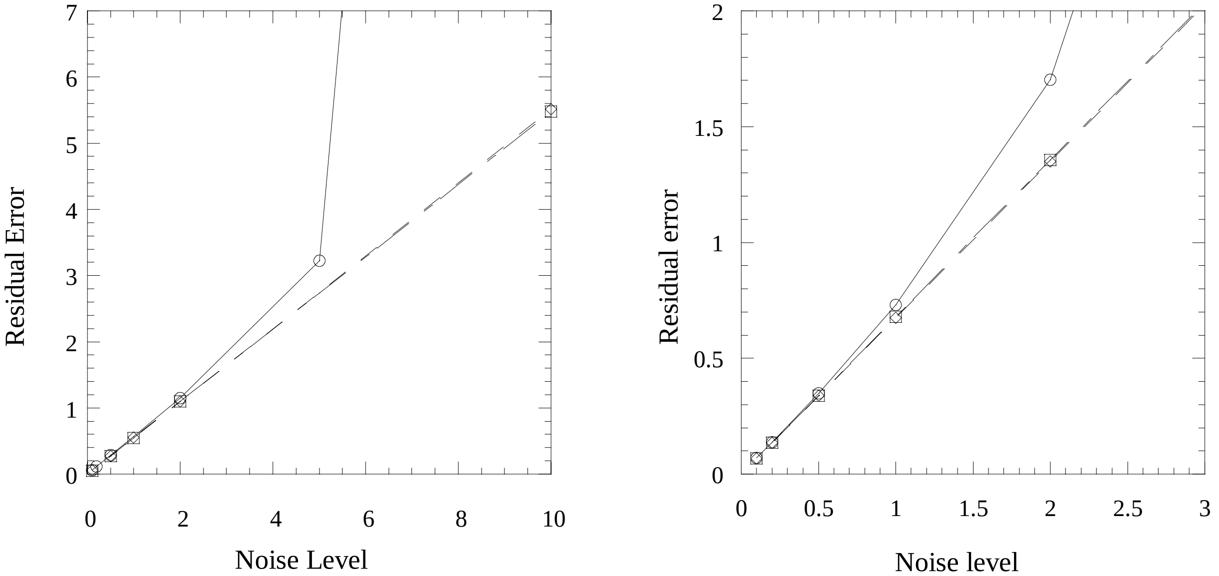
Example: Comparison of Different Methods



a b

Note: solid: DLTn, dashed: reconstruction loss [HZ04, p. 116]

Example: Comparison of Different Methods



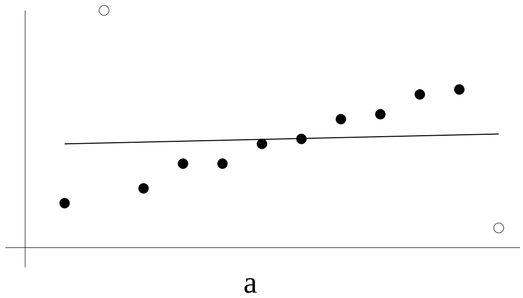
c d

Note: solid: DLTn, dashed: reconstruction loss; c) 10 points, d) 50 points [HZ04, p. 116]

Outline

Outliers and Robust Estimation

- When estimating a transformation from pairs of corresponding points, having these correspondences estimated from data themselves, we expect **noise**: **wrong correspondences**.
- Wrong correspondences could be not just a little bit off, but **way off**: **outliers**.
- Some losses, esp. least squares, are **sensitive to outliers**:



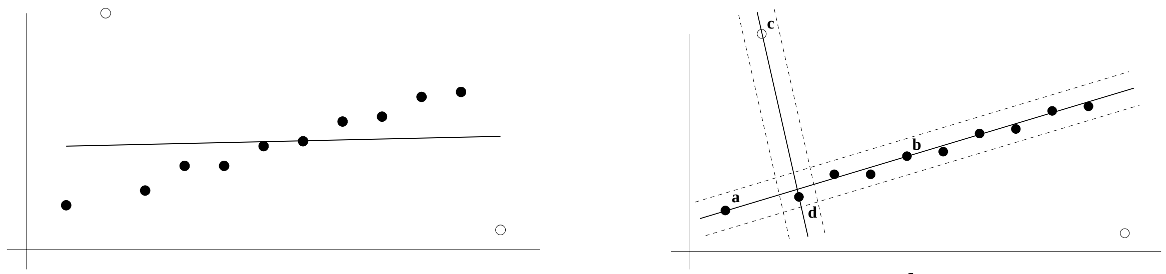
- **Robust estimation:** estimation that is less sensitive to outliers.

[HZ04, p. 117]

Random Sample Consensus (RANSAC)

idea:

1. draw iteratively random samples of data points
   - many and small enough so that some will have no outliers with high probability
2. estimate the model from such a sample
3. grade the samples by the **support** of their models
   - support: number of well-explained points, i.e., points with a small error under the model (inliers)
4. reestimate the model on the support of the best sample



a b

[HZ04, p. 117]

Model Estimation Terminology

- RANSAC works like a wrapper around any estimation method.
- examples:
  - estimating a transformation from point correspondences
  - estimating a line (a linear model) from 2d points
- model estimation terminology:

$$\mathcal{X} \text{ **data space**, e.g. } \mathbb{R}^2$$

$$\mathcal{D} \subseteq \mathcal{X} \text{ **dataset**, e.g. } \mathcal{D} = \{x_1, \ldots, x_N\}$$

$$f(\theta \mid \mathcal{D}) := \frac{1}{|\mathcal{D}|} \sum_{x \in \mathcal{D}} \ell(x, \theta) \text{ **objective**}$$

$$\ell : \mathcal{X} \times \Theta \to \mathbb{R} \text{ **loss/error**, e.g. } \ell(\begin{pmatrix} x \\ y \end{pmatrix}; \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}) := (y - (\theta_1 + \theta_2 x))^2$$

$$\Theta \text{ **(model) parameter space**, e.g. } \mathbb{R}^2$$

$$a : \mathcal{P}(\mathcal{X}) \to \Theta \text{ **estimation method**, e.g. gradient descent}$$

$$\text{aiming at } a(\mathcal{D}) \approx \arg\min_{\theta \in \Theta} f(\theta \mid \mathcal{D})$$

RANSAC Algorithm

procedure
EST-RANSAC($\mathcal{D}, \ell, a;\; N' \in \mathbb{N}, T \in \mathbb{N}, \ell_{\text{max}} \in \mathbb{R}, \text{sup}_{\text{min}} \in \mathbb{N}$)
  $\mathcal{S}_{\text{best}} := \emptyset$
  for $t = 1, \ldots, T$ or until $|\mathcal{S}| \geq \text{sup}_{\text{min}}$ **do**
    $\mathcal{D}' \sim \mathcal{D}$ of size N' ▷ draw a sample
    $\hat{\theta} := a(\mathcal{D}')$ ▷ estimate the model
    $\mathcal{S} := \{x \in \mathcal{D} \mid \ell(x, \hat{\theta}) < \ell_{\text{max}}\}$ ▷ compute support
    if $|\mathcal{S}| > |\mathcal{S}_{\text{best}}|$ **then**
      $\mathcal{S}_{\text{best}} := \mathcal{S}$
  $\hat{\theta} := a(\mathcal{S}_{\text{best}})$ ▷ reestimate the model
  return $\hat{\theta}$

What is a good **sample size N'**?

- often the minimum number to get a unique solution is used.

What is a good **maximal support loss** $\ell_{\max}$?

- for squared distance/L2 loss: $\ell(x, x') := (x - x')^2$
- assume Gaussian noise: $x_{\text{obs}} \sim \mathcal{N}(x_{\text{true}}, \Sigma)$,
  - isotrop noise
  - but no noise in some directions
    - e.g., points on a line: noise only orthogonal to the line

  $\rightsquigarrow \Sigma = USU^T$, $S = \text{diag}(s_1, s_2), s_i \in \{\sigma^2, 0\}, UU^T = I$

$\rightsquigarrow \ell(x_{\text{obs}}, x_{\text{true}}) \sim \sigma^2 \chi^2_m, \quad m := \text{rank}(S)$ degrees of freedom

- inlier: $\ell(x_{\text{obs}}, x_{\text{true}}) < \ell_{\max}$ with probability α
  $\ell_{\max} := \sigma^2 \text{CDF}^{-1}_{\chi^2_m}(\alpha)$

m	model	$\ell_{\max}(\alpha = 0.95)$
1	line, fundamtental matrix	$3.84\sigma^2$
2	projectivity, camera matrix	$5.99\sigma^2$
3	trifocal tensor	$7.81\sigma^2$

What is a good **sample frequency** T?

- find T s.t. at least one of the samples contains no outliers with high probability $\alpha := 0.99$.
- denote $p(x \text{ is an outlier}) = \epsilon$:

$$p(\mathcal{D}' \text{ contains no outliers}) = (1 - \epsilon)^{N'}$$

$$p(\text{at least one } \mathcal{D}' \text{ contains no outliers}) = 1 - (1 - (1 - \epsilon)^{N'})^T \overset{!}{=} \alpha$$

$$\rightsquigarrow T = \frac{1 - \alpha}{1 - (1 - \epsilon)^{N'}}$$

	$\epsilon = p(x \text{ is an outlier})$					
N'	5%	10%	20%	30%	40%	50%
2	2	3	5	7	11	17
3	3	4	7	11	19	35
4	3	5	9	17	34	72
5	4	6	12	26	57	146
6	4	7	16	37	97	293
7	4	8	20	54	163	588
8	5	9	26	78	272	1177

What is a good **sufficient support size sup$_{\text{min}}$**?

- the sufficient support size is an **early stopping criterion**.
- stop if we have as many inliers as expected:

$$\text{sup}_{\text{min}} = N(1-\epsilon)$$

RANSAC Algorithm / Repeated Reestimation

procedure $\text{EST-RANSAC-RERE}(\mathcal{D}, \ell, a;\ \ N' \in \mathbb{N}, T \in \mathbb{N}, \ell_{\text{max}} \in \mathbb{R}, \text{sup}_{\text{min}} \in \mathbb{N})$

$\vdots$

$\quad \mathcal{S} := \mathcal{S}_{\text{best}}$

$\quad$ **do**

$\quad\quad \mathcal{S}_{\text{final}} := \mathcal{S}$

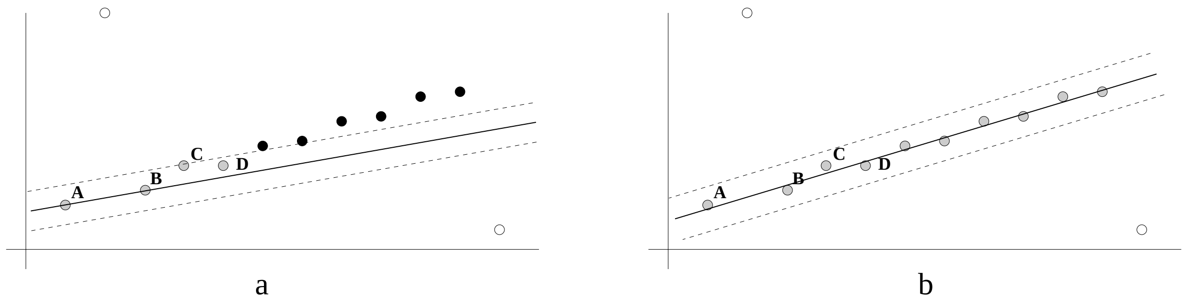
$\quad\quad \hat{\theta} := a(\mathcal{S}_{\text{final}})$ ▷ reestimate the model

$\quad\quad \mathcal{S} := \{x \in \mathcal{D} \mid \ell(x, \hat{\theta}) < \ell_{\text{max}}\}$ ▷ compute support

$\quad$ **while** $\mathcal{S}_{\text{final}} \neq \mathcal{S}$

$\quad$ **return** $\hat{\theta}$

RANSAC: Repeated Reestimation



a) estimation from initial sample

b) reestimation from sample plus support

[HZ04, p. 121]

Outline

Putting it All Together

1. **interest points**:
   compute interest points in each image.
2. **putative matches**:
   compute matching pairs of interest points from their proximity and intensity neighborhood.
3. **simultaneously estimate a projectivity (model) and identify outliers** (robust estimation):
   3.1 estimate a projectivity H from several samples of 4 points and keep the one with maximal support/inliers (**RANSAC** using **DLTn**)
   3.2 reestimate the projectivity H using the best sample and all its support/inliers
   (using **Levenberg-Marquardt**; RANSAC final step)
   3.3 **Guided Matching**: use projectivity H to identify a search region about the transferred points (with relaxed threshold)

Example

Left and right image:


a


b

ca. 500+500 interest points ("corners"):

[HZ04, p. 126]

Summary

- ...

Further Readings

- [HZ04, ch. 4].
- For iterative estimation methods in CV see [HZ04, appendix 6].
- You may also read [HZ04, ch. 5] which will not be covered in the lecture explicitly.

References

Richard Hartley and Andrew Zisserman.
Multiple view geometry in computer vision.
Cambridge university press, 2004.