# Computer Vision

## 1. Projective Geometry in 2D

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# Syllabus

# Outline

1. Very Brief Introduction

2. The Projective Plane

3. Projective Transformations

4. Recovery of Affine Properties from Images

5. Angles in the Projective Plane

6. Recovery of Metric Properties from Images

7. Organizational Stuff

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## 1. Very Brief Introduction

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7. Organizational Stuff

# Topics of the Lecture

1. Simultaneous Localization and Mapping from Video (Visual SLAM)

2. Image Classification and Description

# Simultaneous Localization and Mapping



[source https://www.youtube.com/watch?v=bDOnn0-4Nq8]

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# Simultaneous Localization and Mapping from Video

- SLAM usually employs laser range scanners (lidars).

- **Visual SLAM**: use video sensors (cameras).

- main parts required:
  1. Projective Geometry
  2. Point Correspondences
  3. Estimating Camera Positions (Localization)
  4. Triangulation (Mapping)

# Image Classification and Description



| Describes without errors | Describes with minor errors | Somewhat related to the image | Unrelated to the image |
| --- | --- | --- | --- |

A person riding a motorcycle on a dirt road.

Two dogs play in the grass.

A skateboarder does a trick on a ramp.

A dog is jumping to catch a frisbee.

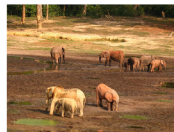A group of young people playing a game of frisbee.

Two hockey players are fighting over the puck.

A little girl in a pink hat is blowing bubbles.

A refrigerator filled with lots of food and drinks.

A herd of elephants walking across a dry grass field.

A close up of a cat laying on a couch.

A red motorcycle parked on the side of the road.

A yellow school bus parked in a parking lot.

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# Outline

# Motivation

In Euclidean (planar) geometry, there are many exceptions, e.g.,

▶ most two lines intersect in exactly one point.

▶ but some two lines do not intersect.

    ▶ parallel lines

# Motivation

In Euclidean (planar) geometry, there are many exceptions, e.g.,

- ► most two lines intersect in exactly one point.
- ► but some two lines do not intersect.
    - ► parallel lines

Idea:

- ► add **ideal points**, one for each set of parallel lines / direction
- ► define these points as intersection of any two parallel lines
- ► now any two lines intersect in exactly one point
    - ► either in a finite or in an ideal point

# Homogeneous Coordinates: Points

Inhomogeneous coordinates:

$$x \in \mathbb{R}^2$$

Homogeneous coordinates:

$$x \in \mathbb{P}^2 := (\mathbb{R}^3 \setminus \{(0,0,0)^T\})/\equiv$$
$$x \equiv y :\Longleftrightarrow \exists s \in \mathbb{R} \setminus \{0\} : sx = y, \quad x, y \in \mathbb{R}^3$$

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Example:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix} \text{ represent the same point in } \mathbb{P}^2$$

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \text{ represents a different point in } \mathbb{P}^2$$

# Homogeneous Coordinates: Points

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**finite points**: $\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} =: \iota(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix})$

**ideal points**: $\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

# Homogeneous Coordinates: Lines

Inhomogeneous coordinates:

$$a \in \mathbb{R}^3 : \ell_a := \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid a_1 x_1 + a_2 x_2 + a_3 = 0 \}$$

- $a_1 \neq 0$ or $a_2 \neq 0$ (or both $a_1, a_2 \neq 0$).
- $sa = (sa_1, sa_2, sa_3)^T$ encodes the same line as $a$ (any $s \in \mathbb{R}, s \neq 0$).

Note: $\kappa : \mathbb{R}^3 \setminus \{(0,0,0)^T\} \to \mathbb{P}^2, a \mapsto [a] := \{a' \in \mathbb{R}^3 \mid a' \equiv a\}$.

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Homogeneous coordinates:

$$a \in \mathbb{P}^2 : \ell_a := \{ x \in \mathbb{P}^2 \mid a^T x = a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \}$$

- contains all finite points of line $a' \in \kappa^{-1}(a)$: $\ell_{\kappa(a')} \supsetneqq \iota(\ell_{a'})$
- and the ideal point $(a_2, -a_1, 0)^T$.
  - intersection of parallel lines (same $a_1, a_2$, different $a_3$)

Note: $\kappa : \mathbb{R}^3 \setminus \{(0,0,0)^T\} \to \mathbb{P}^2, a \mapsto [a] := \{ a' \in \mathbb{R}^3 \mid a' \equiv a \}$.

# A point on a line

A point $x$ lies on line $a$ iff $x^T a = 0$.

# Intersection of two lines

Lines $a$ and $b$ intersect in $a \times b := \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

Proof:

$a^T (a \times b) = a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0$

$b^T (a \times b) = \ldots = 0$

## Intersection of two lines

Lines $a$ and $b$ intersect in $a \times b := \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

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$$b^T (a \times b) = \ldots = 0$$

Example:

$$x = 1 : \ a = (-1, 0, 1)^T$$
$$y = 1 : \ b = (0, -1, 1)^T$$
$$a \times b = (1, 1, 1)^T$$

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$$b^T(a \times b) = \ldots = 0$$

Esp. for parallel lines: $b_1 = a_1, b_2 = a_2, b_3 \neq a_3$:

$$a \times b \equiv \begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix}$$

# Line joining points

The line through $x$ and $y$ is $x \times y$.

Proof: exactly the same as previous slide.

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Example:

$$x = (-1, 0, 1)^T$$
$$y = (0, -1, 1)^T$$
$$x \times y = (1, 1, 1)^T$$

# Line at infinity

All ideal points form a line:

$$l_\infty := (0, 0, 1)^T \quad \textbf{line at infinity}$$

Proof:

for any ideal point $x = (x_1, x_2, 0)^T$: $x^T l_\infty = 0$.

for any finite (real-valued) point $x = (x_1, x_2, 1)$: $x^T l_\infty = 1 \neq 0$.

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for any finite (real-valued) point $x = (x_1, x_2, 1)$: $x^T l_\infty = 1 \neq 0$.

Furthermore:

► This is the only line in $\mathbb{P}^2$ **not** corresponding to an Euclidean line.

► Two parallel lines meet at the line at infinity.

# A model for the projective plane
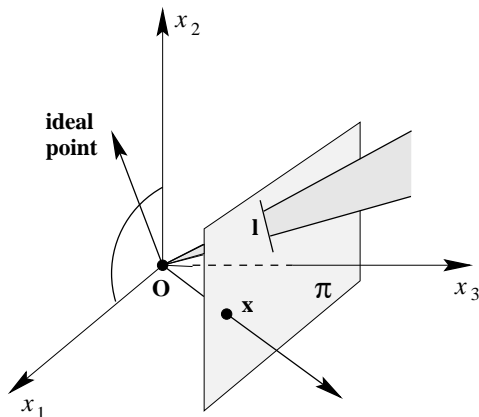


- points correspond to rays (lines through the origin)
- lines correspond to planes through the origin. [HZ04, p. 29]

# Conics

▶ A **conic section** (or just **conic**) is a curve one gets as intersection of a cone and a plane
  ▶ ellipsis, parabola, hyperbola
▶ Corresponds to a curve of degree 2:
  Heterogeneous coordinates:

$$a \in \mathbb{R}^6 : \mathbf{C}_a := \{x \in \mathbb{R}^2 \mid a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1 + a_5 x_2 + a_6 = 0\}$$
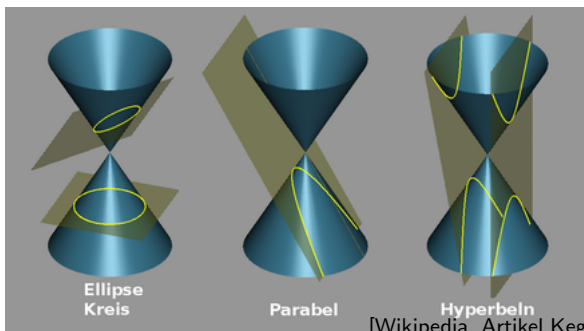


[Wikipedia, Artikel Kegelschnitt]

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Homogeneous coordinates:

$$a \in \mathbb{P}^5 : \mathbf{C}_a := \{x \in \mathbb{P}^2 \mid a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2$$
$$+ a_4 x_1 x_3 + a_5 x_2 x_3 + a_6 x_3^2 = 0\}$$

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$$+ a_4 x_1 x_3 + a_5 x_2 x_3 + a_6 x_3^2 = 0\}$$
$$= \{x \in \mathbb{P}^2 \mid x^T C x = 0\}, C := \begin{pmatrix} a_1 & a_2/2 & a_4/2 \\ a_2/2 & a_3 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix}$$

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Homogeneous coordinates:

$$C \in \text{Sym}(\mathbb{P}^{3\times 3}) : \mathbf{C}_C := \{x \in \mathbb{P}^2 \mid x^T C x = 0\}$$

# A conic joining 5 points

- Let $x^1, \ldots, x^5 \in \mathbb{P}^2$ be 5 points
  - in general position (i.e., never more than 2 on the same line)
- Conic parameters $a$ have to fulfil the following system of linear equations:

$$\begin{pmatrix} x_1^1 x_1^1 & x_1^1 x_2^1 & x_2^1 x_2^1 & x_1^1 x_3^1 & x_2^1 x_3^1 & x_3^1 x_3^1 \\ x_1^2 x_1^2 & x_1^2 x_2^2 & x_2^2 x_2^2 & x_1^2 x_3^2 & x_2^2 x_3^2 & x_3^2 x_3^2 \\ x_1^3 x_1^3 & x_1^3 x_2^3 & x_2^3 x_2^3 & x_1^3 x_3^3 & x_2^3 x_3^3 & x_3^3 x_3^3 \\ x_1^4 x_1^4 & x_1^4 x_2^4 & x_2^4 x_2^4 & x_1^4 x_3^4 & x_2^4 x_3^4 & x_3^4 x_3^4 \\ x_1^5 x_1^5 & x_1^5 x_2^5 & x_2^5 x_2^5 & x_1^5 x_3^5 & x_2^5 x_3^4 & x_3^5 x_3^5 \end{pmatrix} a = 0$$

# Degenerate Conics

Conic $C$ **degenerate**: $C$ does not have full rank.

Example: two lines $C := ab^T + ba^T$ (rank 2).

▶ contains lines $a$ and $b$.
  proof: for points $x$ on line $a$: $x^T a = 0$.
    $\rightsquigarrow$ $x$ also on $C$: $x^T C x = x^T a b^T x + x^T b a^T x = 0$.

# Conic tangent lines

The tangent line to a conic $C$ at a point $x$ is $Cx$.

Proof:

x lies on $Cx$: $x^T Cx = 0$.

If there is another common point $y$: $y^T Cy = 0$ and $y^T Cx = 0$.

$\rightsquigarrow x + \alpha y$ is common for all $\alpha$, i.e., the whole line.

$\rightsquigarrow C$ is degenerate (or there is no such $y$).

# Outline

# Projectivity

A map $h : \mathbb{P}^2 \to \mathbb{P}^2$ is called **projectivity**, if

1. it is invertible and
2. it preserves lines,
   i.e., whenever $x, y, z$ are on a line, so are $h(x), h(y), h(z)$.

Equivalently, $h(x) := Hx$ for a non-singular $H \in \mathbb{P}^{3 \times 3}$.

Note: $H^{-T} := (H^{-1})^T$.

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Equivalently, $h(x) := Hx$ for a non-singular $H \in \mathbb{P}^{3 \times 3}$.

Proof:
Any map $h(x) := Hx$ is a projectivity:
  Let $x$ be a point on line $a$: $a^T x = 0$.
  Then point $Hx$ is on line $H^{-T} a$: $(H^{-1} a)^T Hx = a^T H^{-1} Hx = a^T x = 0$.

Any projectivity $h$ is of type $h(x) = Hx$: more difficult to show.

Note: $H^{-T} := (H^{-1})^T$.

## Transformation of Lines and Conics

The image of a line $a$ under projectivity $H$ is the line $H^{-T}a$:

$$H(l_a) = l_{H^{-T}a}$$

Proof:

Let $x$ be a point on line $a$: $a^T x = 0$.
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Then point $Hx$ is on line $H^{-1}a$: $(H^{-T}a)^T Hx = a^T H^{-1}Hx = a^T x = 0$.

The image of a conic $C$ under projectivity $H$ is the conic $H^{-T}CH^{-1}$:

$$H(\mathbf{C}_C) = \mathbf{C}_{H^{-T}CH^{-1}}$$

Proof:

Let $x$ be a point on conic $C$: $x^T C x = 0$.
Then point $Hx$ is on conic $H^{-T}CH^{-1}$: $x^T H^T H^{-T} C H^{-1} Hx = 0$

# A Hierarchy of Transformations

▶ The projective transformations form a group (**projective linear group**:

$$\mathsf{PL}_n := \mathsf{GL}_n/ \equiv\; = \{H \in \mathbb{P}^{3\times 3} \mid H \text{ invertible}\}$$

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- ▶ There are several subgroups:
    - ▶ **affine group**: last row is $(0, 0, 1)$
    - ▶ **Euclidean group**: additionally $H_{1:2,1:2}$ orthogonal
    - ▶ **oriented Euclidean group**: additionally $\det H = 1$

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  - **oriented Euclidean group**: additionally $\det H = 1$

- These subgroups can be described two ways:
  - structurally (as above)
  - by **invariants**: objects or sets of objects mapped to themselves

# Isometries

$$\begin{pmatrix} x_1' \\ x_2' \\ 1 \end{pmatrix} = \begin{pmatrix} \epsilon\cos\theta & -\sin\theta & t_1 \\ \epsilon\sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} x$$

- **rotation matrix** $R$: $R^T R = R R^T = I$
- **translation vector** $t$.
- **orientation preserving** if $\epsilon = +1$ (equivalent to $\det R = +1$)
  ($\epsilon \in \{+1, -1\}$)

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Invariants:
- length, angle, area
- line at infinity $l_\infty$

# Similarity Transformations

$$
\begin{pmatrix} x'_1 \\ x'_2 \\ 1 \end{pmatrix} = \begin{pmatrix} s\cos\theta & -s\sin\theta & t_1 \\ s\sin\theta & s\cos\theta & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} x
$$

▶ **isotropic scaling $s$.**

# Similarity Transformations

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- **isotropic scaling $s$.**

Invariants:

- angle
- ratio of lengths, ratio of areas
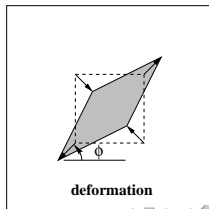- line at infinity $l_\infty$

# Affine Transformations

$$\begin{pmatrix} x_1' \\ x_2' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & t_1 \\ a_{2,1} & a_{2,2} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} x$$
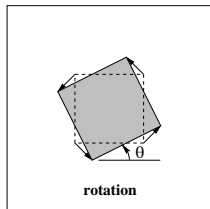
▶ $A$ non-singular

# Affine Transformations

$$\begin{pmatrix} x_1' \\ x_2' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & t_1 \\ a_{2,1} & a_{2,2} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} x$$

▶ $A$ non-singular, decompose via SVD:

$$A = R(\theta)R(-\phi) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R(\phi)$$

▶ **non-isotropic scaling** with axis $\phi$



| rotation | deformation |

[HZ04, p. 40]

# Affine Transformations

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▶ **non-isotropic scaling** with axis $\phi$

Invariants:
  ▶ parallel lines
  ▶ ratio of lengths of parallel line segments
  ▶ ratio of areas
  ▶ line at infinity $l_\infty$

# Projective Transformations

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & t_1 \\ a_{2,1} & a_{2,2} & t_2 \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A & t \\ v^T & v_3 \end{pmatrix} x$$

- **$v$ moves the line at infinity $l_\infty$**
  - $l_v \mapsto l_\infty$
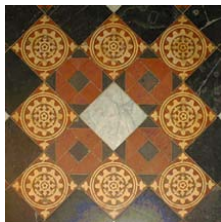
# Projective Transformations

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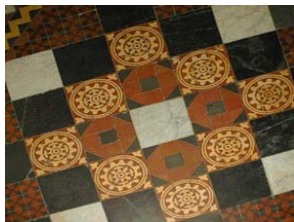
Invariants:

- ratio of ratios of lengths of parallel line segments (**cross ratio**)

# Similary, Affine & Projective Transformations / Example
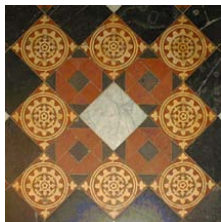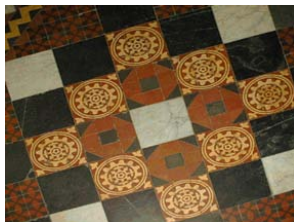


a                          b                          c

| original | image under<br>a) similarity |
|----------|------------------------------|
| circles | circles |
| squares | squares |
| parallel lines | parallel |
| orthogonal lines | orthogonal |

[HZ04, p. 37]

# Similary, Affine & Projective Transformations / Example



a                              b                              c

| original | image under | |
| | a) similarity | b) affine |
|---|---|---|
| circles | circles | ellipsis |
| squares | squares | diamond |
| parallel lines | parallel | parallel |
| orthogonal lines | orthogonal | non-orthogonal |

[HZ04, p. 37]

# Similary, Affine & Projective Transformations / Example



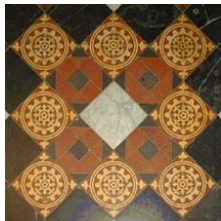a                                  b                                  c
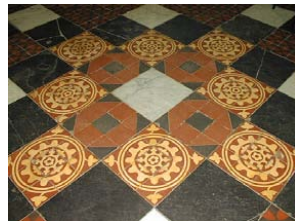
| original | image under | | |
| --- | --- | --- | --- |
| | a) similarity | b) affine | c) projective |
| circles | circles | ellipsis | conic |
| squares | squares | diamond | quadrangle |
| parallel lines | parallel | parallel | converging |
| orthogonal lines | orthogonal | non-orthogonal | non-orthogonal |

[HZ04, p. 37]
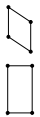
# Projective Transformations / Decomposition

$$\left( \begin{array}{cc} A & t \\ v^T & v_3 \end{array} \right) = \left( \begin{array}{cc} sR & t \\ 0^T & 1 \end{array} \right) \left( \begin{array}{cc} K & 0 \\ 0^T & 1 \end{array} \right) \left( \begin{array}{cc} I & 0 \\ v^T & v_3 \end{array} \right)$$

$$A = sRK + tv^T$$

- $K$ upper triangular matrix with $\det K = 1$
- valid for $v_3 \neq 0$
- unique if $s$ is chosen $s > 0$

# Summary of Projective Transformations

| Group | Matrix | Distortion | Invariant properties |
|---|---|---|---|
| Projective 8 dof | $\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$ | | Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths). |
| Affine 6 dof | $\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ | | Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$. |
| Similarity 4 dof | $\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ | | Ratio of lengths, angle. The circular points, $\mathbf{I}$, $\mathbf{J}$ (see section 2.7.3). |
| Euclidean 3 dof | $\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ | | Length, area |

[HZ04, p. 44]

Lars Schmidt-Thieme, Information Systems and Machine Learning Lab (ISMLL), University of Hildesheim, Germany

# Outline

# Recovery of Affine and Metric Properties

Decomposition of general projective transformation:

$$\begin{pmatrix} A & t \\ v^T & v_3 \end{pmatrix} = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix}$$

1. undo proper projective transformation (**affine rectification**):
   - then original and image differ only by an affine transformation
   - $\rightsquigarrow$ measure **affine properties** of the original in the image
     (= properties invariant under affine transformations)
     - parallel lines, ratio of lengths on parallel lines

# Recovery of Affine and Metric Properties

Decomposition of general projective transformation:

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1. undo proper projective transformation (**affine rectification**):
   - then original and image differ only by an affine transformation
   - ⇝ measure **affine properties** of the original in the image
     (= properties invariant under affine transformations)
       - parallel lines, ratio of lengths on parallel lines

2. undo proper affine transformation (**metric rectification**):
   - then original and image differ only by a similarity transformation
   - ⇝ measure **metric properties** of the original in the image
     (= properties invariant under similarity transformations)
       - angles, ratio of lengths

# Recovery of Affine Properties

Undo proper projective transformation:

$$
\begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ v_1 x_1 + v_2 x_2 \end{pmatrix}
$$

$$
l_\infty := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -v/v_3 \\ 1/v_3 \end{pmatrix} = \frac{1}{v_3} \begin{pmatrix} -v_1 \\ -v_2 \\ 1 \end{pmatrix}
$$

▶ maps line at infinity to finite line $(-v_1, -v_2, 1)^T$
▶ to undo:
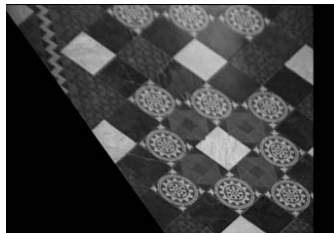  ▶ locate image $(-v_1, -v_2, 1)^T$ of line at infinity
  ▶ undo by applying the inverse $H^{-1} = \begin{pmatrix} I & 0 \\ -v^T/v_3 & 1/v_3 \end{pmatrix}$

Note: Lines transform by $H^{-T}$: $\begin{pmatrix} I & 0 \\ v^T & v_3 \end{pmatrix}^{-T} = \begin{pmatrix} I & -v/v_3 \\ 0 & 1/v_3 \end{pmatrix}$.
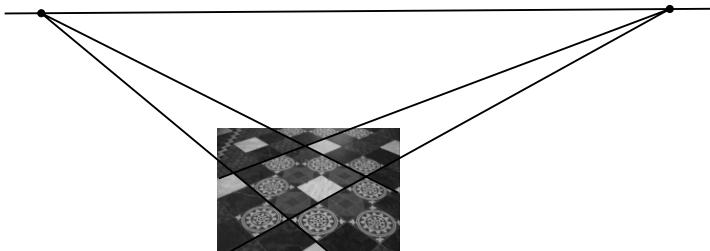
# Recovery of Affine Properties / Example



a

b

[HZ04, p. 50]

Now we can measure area ratios !

# Recovery of Affine Properties / Algorithm

1: **procedure** RECTIFY-AFFINE-TWO-PARALLELS($a^1, a^2, b^1, b^2 \in \mathbb{P}^2$)
2:      $s^1 := a^1 \times a^2$      $\triangleright$ compute intersection of parallels $a^1, a^2$
3:      $s^2 := b^1 \times b^2$      $\triangleright$ compute intersection of parallels $b^1, b^2$
4:      $l_\infty := s^1 \times s^2$      $\triangleright$ compute image of line at infinity
5:      $H^{-1} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{\infty,1}/l_{\infty,3} & -l_{\infty,2}/l_{\infty,3} & 1/l_{\infty,3} \end{pmatrix}$      $\triangleright$ compute inverse
6:      **return** $H^{-1}$

# Outline

# Circular Points

A conic

$$
C := \begin{pmatrix} a_1 & a_2/2 & a_4/2 \\ a_2/2 & a_3 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & a_4/2 \\ 0 & a_1 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix}
$$

is a circle if $a_1 = a_3$ and $a_2 = 0$.

# Circular Points

A conic

$$C := \begin{pmatrix} a_1 & a_2/2 & a_4/2 \\ a_2/2 & a_3 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & a_4/2 \\ 0 & a_1 & a_5/2 \\ a_4/2 & a_5/2 & a_6 \end{pmatrix}$$

is a circle if $a_1 = a_3$ and $a_2 = 0$.

Ideal points $x = (x_1, x_2, 0)^T$ on a circle:

$$x^T C x = a_1 x_1^2 + a_1 x_2^2 = 0$$

are exactly the **circular points**:

$$I := \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad J := \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

# Line Conics

$C \in \text{Sym}(\mathbb{P}^{3 \times 3})$ defines a **point conic** via

$$\mathbf{C}_C := \{x \in \mathbb{P}^2 \mid x^T C x = 0\}$$

It also can be used to define a **line conic** / **dual conic**:

$$\mathbf{C}_C^* := \{a \in \mathbb{P}^2 \mid a^T C a = 0\}$$

(where $a$ denotes a line)

# Adjugate of a Matrix

For a square matrix $A \in \mathbb{R}^{n \times n}$,

$$A^* \in \mathbb{R}^{n \times n} \text{ with } A_{i,j}^* := (-1)^{i+j} \det A_{-j,-i}$$

is called its **adjugate** $A^*$.

It holds:

- for any $A$: $A^*A = AA^* = (\det A)I$
- $A^*$ is continuous in $A$.
- if $A$ is invertible, the adjugate is the scaled inverse: $A^* = (\det A)A^{-1}$
- if $A$ is not invertible, the adjugate nullifies $A$: $A^*A = AA^* = 0$
- the adjugate is the transposed of the cofactor matrix.

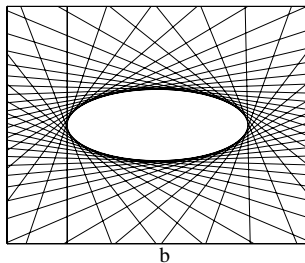Note: $A_{-j,-i}$ denotes the matrix $A$ with row $j$ and column $i$ removed.
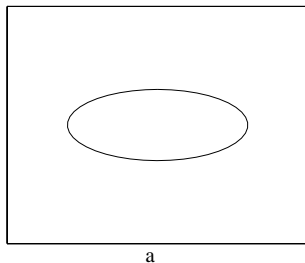The adjugate is also called **adjoint**.

## Dual Conic

For any point conic $C \in \mathrm{Sym}(\mathbb{P}^{3\times3})$, the set of tangent lines

- forms a line conic,
- parametrized by the adjugate $C^*$:

$$\{a \in \mathbb{P}^2 \mid a \text{ tangent to } C\} = \mathbf{C}_{C^*}^*$$



a



b

[HZ04, p. 32]

# Dual Conic to the Circular Points

**Dual conic to the circular points** (degenerate):

$$C_\infty^* := IJ^T + JI^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- ▶ contains exactly all lines through the circular points $I$ or $J$.
- ▶ transforms as $HC^*H^T$: $H(\mathbf{C}_{C^*}^*) = \mathbf{C}_{HC^*H^T}^*$.
- ▶ fixed under projectivity $H$ iff $H$ is a similarity.
- ▶ 4 dof (general $C$ has 5, minus 1 due to $\det C = 0$)
- ▶ $l_\infty$ is the null vector of $C_\infty^*$.

# Angels in the Projective Plane

Angels are defined as:

$$\cos\theta(a,b) := \frac{a^T C_\infty^* b}{\sqrt{(a^T C_\infty^* a)\,(b^T C_\infty^* b)}}, \quad a, b \in \mathbb{P}^2$$

- for the canonical $C_\infty^*$, conincides with the Euclidean definition:

$$\cos\theta(a,b) := \frac{a^T b}{\sqrt{(a^T a)\,(b^T b)}}, \quad a, b \in \mathbb{R}^2$$

- stays invariant under projective transformation:

$$a' = H^{-T} a, \quad b' = H^{-T} b, \quad C_\infty^{*\,\prime} = H C_\infty^* H^T$$

$$a'^T C_\infty^{*\,\prime} b' = a^T H^{-1} H C_\infty^* H^T H^{-T} b = a^T C_\infty^* b$$

# Outline

# Recovery of Metric Properties

- ▶ assume there is no pure projective transformation
  (i.e., affine rectification already done).
- ▶ need only to find pure affine transformation:

$$H_a := \left( \begin{array}{cc} K & 0 \\ 0^T & 1 \end{array} \right), \quad \text{with } K \text{ upper triangular}$$

- ▶ under $H_a$ we get $C_\infty^* {}'$ as

$$C_\infty^* {}' := H_a C_\infty^* H_a^T = \left( \begin{array}{cc} KK^T & 0 \\ 0^T & 0 \end{array} \right)$$

  1. find symmetric matrix $S := KK^T$
  2. find $K$ via Cholesky decomposition of $S$

# Recovery of Metric Properties (2/2)

▶ for two lines $a', b'$ that are orthogonal in the original:

$$
\begin{aligned}
0 = a'^T C_\infty^{*\prime} b' &= a'^T_{1:2} S b_{1:2} \\
&= a'_1 S_{1,1} b'_1 + a'_1 S_{1,2} b'_2 + a'_2 S_{2,1} b'_1 + a'_2 S_{2,2} b'_2 \\
&= a'_1 b'_1 S_{1,1} + (a'_1 b'_2 + a'_2 b'_1) S_{1,2} + a'_2 b'_2 S_{2,2} \\
&= (a'_1 b'_1, a'_1 b'_2 + a'_2 b'_1, a'_2 b'_2)(S_{1,1}, S_{1,2}, S_{2,2})^T
\end{aligned}
$$

we get 1 linear constraint in $s := (S_{1,1}, S_{1,2}, S_{2,2})^T$.

▶ for two pairs of lines that are orthogonal in the original we get
2 linear constraints for 3 variables

$$
\begin{pmatrix}
a'_1 b'_1 & a'_1 b'_2 + a'_2 b'_1 & a'_2 b'_2 \\
c'_1 d'_1 & c'_1 d'_2 + c'_2 d'_1 & c'_2 d'_2
\end{pmatrix} s
$$

where $s \neq 0$ has to be identified only up to a factor.

# Recovery of Metric Properties / Algorithm

1: **procedure**
   RECTIFY-METRIC-TWO-ORTHOGONALS($a^1, a^2, b^1, b^2 \in \mathbb{P}^2$)

2: $\quad A := \begin{pmatrix} a_1^1 a_1^2 & a_1^1 a_2^2 + a_2^1 a_1^2 & a_2^1 a_2^2 \\ b_1^1 b_1^2 & b_1^1 b_2^2 + b_2^1 b_1^2 & b_2^1 b_2^2 \end{pmatrix}$
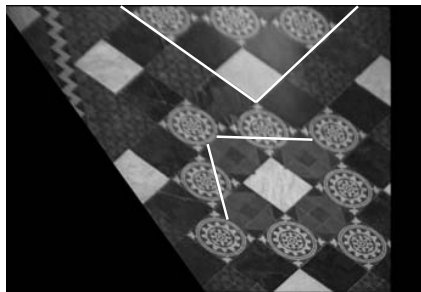
3: $\quad$ find $s \neq 0 : As = 0$ $\qquad\qquad\qquad$ $\triangleright$ find $C_\infty^* := \begin{pmatrix} s_1 & s_2 & 0 \\ s_2 & s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4: $\quad K := \mathsf{cholesky}\left(\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}\right)$ $\qquad$ $\triangleright$ find $H := \begin{pmatrix} K & 0 \\ 0^T & 1 \end{pmatrix}$

5: $\quad H^{-1} := \begin{pmatrix} 1/K_{1,1} & -1/(K_{1,2}K_{2,2}) & 0 \\ 0 & 1/K_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad$ $\triangleright$ compute inverse
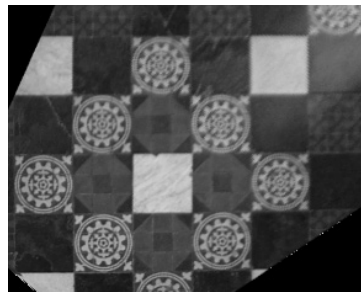
6: $\quad$ **return** $H^{-1}$

# Recovery of Metric Properties / Example



a) affine rectified image

b) metric rectified image

Now we can measure angles and length ratios !

[HZ04, p. 57]

# Outline

## 7. Organizational Stuff

# Exercises and Tutorials

▶ There will be a weekly sheet with 2 exercises
handed out **each Monday** in the lecture.
1st sheet will be handed out today, Mon. 10.4.

▶ Solutions to the exercises can be
submitted until **next Monday 8:00 am**
1st sheet is due Mon. 17.4., 8:00 am

▶ Exercises will be corrected.

▶ Tutorials **each Wednesday 4pm–6pm**,
1st tutorial at Wed. 12.4.

▶ Successful participation in the tutorial gives up to 10% bonus points
for the exam.

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# Exam and Credit Points

- There will be a written exam at end of term
  (2h, 4 problems).

- The course gives 6 ECTS (2+2 SWS).

- The course can be used in
  - International Master in Data Analytics
  - Angewandte Informatik MSc. / Informatik / Gebiet KI & ML
  - IMIT MSc. / Informatik / Gebiet KI & ML
  - Wirtschaftsinformatik MSc / Business Intelligence
  - as well as in both BSc programs.

## Some Text Books

- Richard Hartley, Andrew Zisserman (2004):
  *Multiple View Geometry in Computer Vision*,
  Cambridge University Press.

- Simon J. D. Prince (2012):
  *Computer Vision: Models, Learning, and Inference,*
  Cambridge University Press.

- Richard Szeliski (2011):
  *Computer Vision, Algorithms and Applications*,
  Springer.

- David A. Forsyth, Jean Ponce ($^2$2012, 2007):
  *Computer Vision, A Modern Approach,*
  Prentice Hall.

# Some First Computer Vision Software

- ▶ Open Computer Vision Library (OpenCV)
    - ▶ C++ library
    - ▶ has wrappers for Python & Octave
    - ▶ originally developed by Intel
    - ▶ v3.2, 12/2016; http://opencv.org

Public data sets:

- ▶ ...

# Summary (1/3)

- The **projective plane** $\mathbb{P}^2$ is an extension of the Euclidean plane with **ideal points**.

- Points and lines in $\mathbb{P}^2$ are parametrized by **homogenuous coordinates**.

- Each two parallels intersect in an ideal point, all ideal points form the **line at infinity** $l_\infty$.

- Each circle contains two ideal points, the **circular points**, all lines through the circular points form the **dual conic to the circular points** $C_\infty^*$.

- **Conics** are curves of order 2 (hyperbolas, parabolas, ellipsis), parametrized by a symmetric matrix $C$ containing all points $x$ with $x^T C x = 0$.

# Summary (2/3)

- **Projectivities** $H$ are invertibles mappings of $\mathbb{P}^2$ onto $\mathbb{P}^2$ that preserve lines.

- Lines $a$ transform via $H^{-T}a$, conics $C$ via $H^{-T}CH^{-1}$.

- There exist several subgroups of the group of projectivities:
  - **Isometries rotate** and **translate** figures.
    - preserving lengths
  - **Similarities** additionally **(isotropic) scale** figures.
    - preserving ratio of lengths, angles
  - **Affine transforms** additionally **non-isotropic scale** figures.
    - preserving ratio of lengths on parallel lines, parallel lines
  - **Projectivities** additionally **move the line at infinity**.
    - preserving cross ratios

- Any projectivity can be decomposed into a chain of
  a pure projective, a pure affine transform and a similarity.

# Summary (3/3)

- ▶ Images distorted by an projective transformation can be **rectified** (i.e., the projective transformation undone).

- ▶ **Affine rectification**
  - ▶ undoes a proper projective transformation
  - ▶ moves the **line at infinity** back to its **canonical position**.
  - ▶ allows to measure **affine properties**:
    - ▶ ratio of lengths on parallel lines, parallel lines
  - ▶ requires, e.g., **two pairs of parallel lines**.

- ▶ **Metric rectification**
  - ▶ undoes a proper affine transformation
  - ▶ moves the **dual conic to the circular points** back to its canonical position.
  - ▶ allows to measure **metric properties**:
    - ▶ angles, ratio of lengths
  - ▶ requires, e.g., two pairs of orthogonal lines.

# Further Readings

- [HZ04, ch. 1 and 2].

# References

Richard Hartley and Andrew Zisserman.
*Multiple view geometry in computer vision*.
Cambridge university press, 2004.