Image Analysis

4. Wavelets

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Course on Image Analysis, winter term 2008 1/36

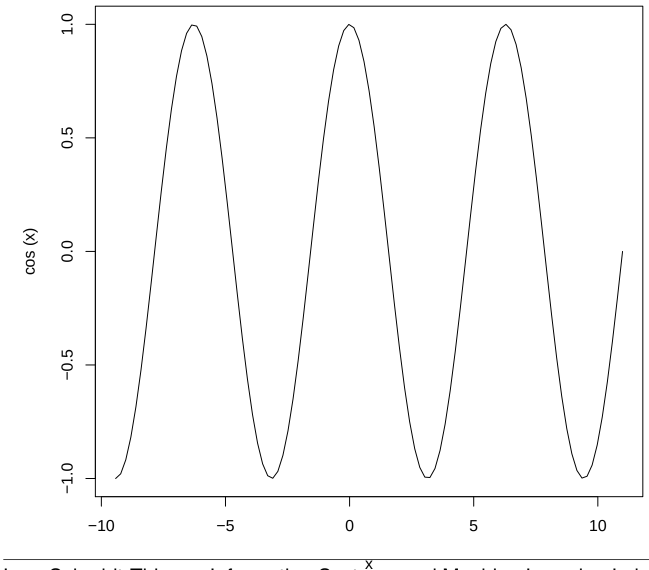
Basis Functions

Fourier Analysis:

$$\psi(x) := \cos x$$

Wavelets:

$$\psi(x) := \text{haar}(x) := \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ -1, & x \in [\frac{1}{2}, 1) \\ 0, & \text{else} \end{cases}$$



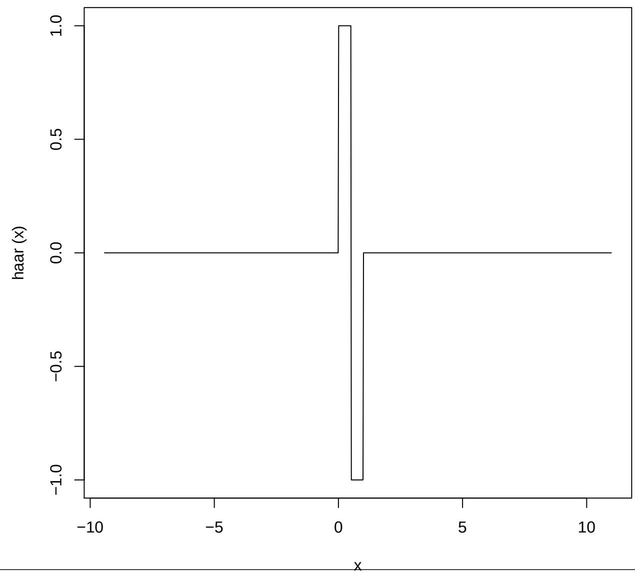


Image Analysis / 1. Haar Wavelets

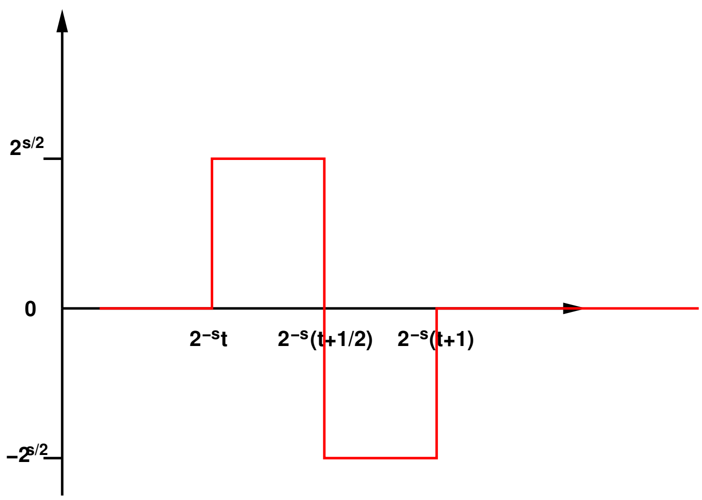
Basis Functions

Fourier Analysis:

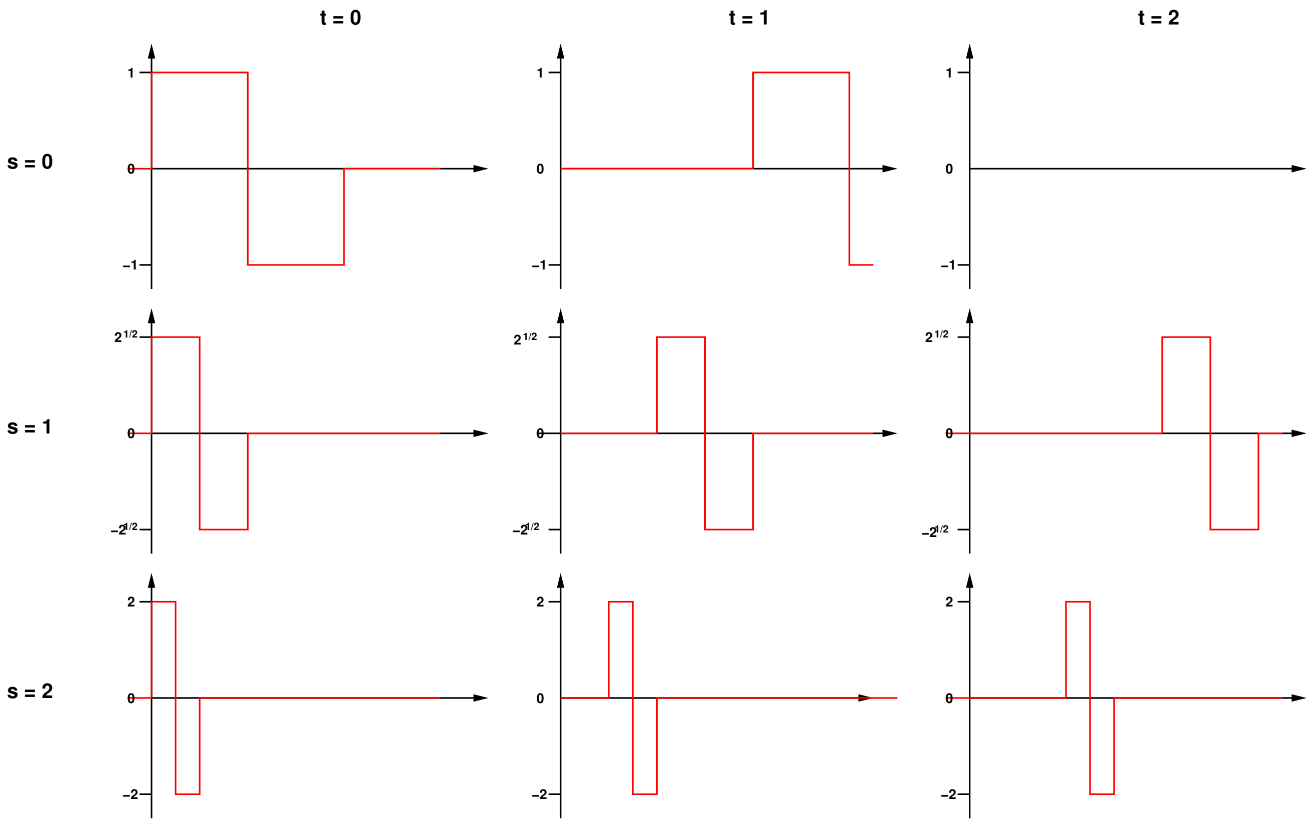
$$\psi_\omega(x) := \cos 2\pi\omega x$$

Wavelets:

$$\begin{aligned}\psi_{s,t}(x) :=& \sqrt{2^s} \cdot \text{haar}(2^s x - t) \\ =& \sqrt{2^s} \cdot \begin{cases} 1, & x \in (2^{-s}t, 2^{-s}(t+\frac{1}{2})) \\ -1, & x \in [2^{-s}(t+\frac{1}{2}), 2^{-s}(t+1)) \\ 0, & \text{else} \end{cases}\end{aligned}$$



Basis Functions



Orthogonality of Basis Functions

Obviously, two distinct Haar basis functions $\psi_{s,t}$ and $\psi_{s',t'}$ with $s, t, s', t' \in \mathbb{Z}$ are orthogonal:

$$\langle \psi_{s,t}, \psi_{s',t'} \rangle := \int_{-\infty}^{\infty} \psi_{s,t}(x) \cdot \psi_{s',t'}(x) \, dx = 0$$

And

$$\langle \psi_{s,t}, \psi_{s,t} \rangle = 1$$

Proof.

If they have the same scale ($s = s'$), then their support does not overlap.

If they have different scale, say $s > s'$, then $\psi_{s,t}$ is constant on the support of $\psi_{s',t'}$, i.e., the integral averages to zero.

$\langle \psi_{s,t}, \psi_{s,t} \rangle$ integrates $\sqrt{2^s} \cdot \sqrt{2^2} = 2^s$ over the support 2^{-s}.

Wavelet Representation

Theorem (Wavelet Representation). Let $\psi_{s,t},\ s,t \in \mathbb{Z}$ be a set of Wavelet basis functions.

Every function $f : \mathbb{R} \rightarrow \mathbb{R}$ (satisfying some regularity conditions) can be written as

$$f(x) = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{s,t}\, \psi_{s,t}(x)$$

with coefficients $c_{s,t} \in \mathbb{R}$.

The coefficients $c_{s,t}$ can be computed as follows:

$$c_{s,t} = \int_{-\infty}^{\infty} f(x) \psi_{s,t}(x) dx$$

Haar Wavelet Representation

For the Haar basis functions this yields

$$f(x) = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{s,t} \cdot \sqrt{2^s}\, \text{haar}(2^s\, x - t)$$

and

$$c_{s,t} = \sqrt{2^s} \left(\int_{2^{-s}\, t}^{2^{-s}\,(t+\frac{1}{2})} f(x)\, dx - \int_{2^s\,(t+\frac{1}{2})}^{2^{-s}\,(t+1)} f(x)\, dx \right)$$

Haar Wavelets / Computing Coefficients

The values of integrals with a simple rectangle impulse on different scales can be computed recursively:

$$a_{s,t} := \sqrt{2^s} \int_{2^{-s}\,t}^{2^{-s}\,(t+1)} f(x)\,dx$$
$$a_{s,t} = \frac{1}{\sqrt{2}} (a_{s+1,2t} + a_{s+1,2t+1})$$

The coefficients of the Haar wavelet can be computed from these values via

$$c_{s,t} = \frac{1}{\sqrt{2}} (a_{s+1,2t} - a_{s+1,2t+1})$$

Haar Wavelets / Discrete Wavelet Transform

For a finite discrete signal f of length 2^n the function can already be represented by a finite sum of Haar wavelets:

$$f(x) = a_{-n,0} + \sum_{s=-n}^{-1} \sum_{t=0}^{2^{n+s}-1} c_{s,t} \cdot \sqrt{2^s}\,\text{haar}(2^s\,x - t)$$

i.e., a composition of Haar wavelets with supports 2, 4, 8 etc.

The initial a values are just the signal values:

$$a_{s=0,t} := \int_{2^{-s}\,t}^{2^{-s}\,(t+1)} f(x)\,dx$$
$$= \int_{t}^{t+1} f(x)\,dx$$
$$= \sum_{x=t}^{<t+1} f(x) = f(t)$$

Haar Wavelets / Computing Coefficients / Example

Let

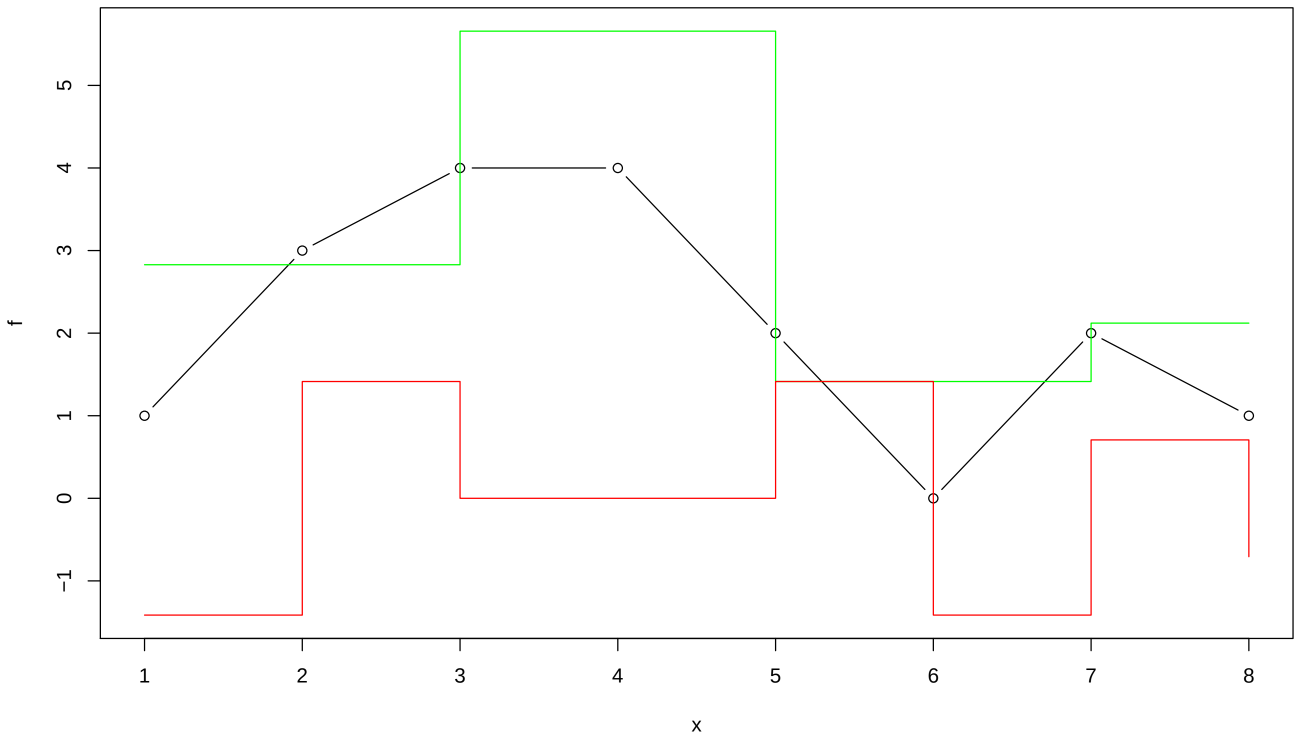
$$f = (1, 3, 4, 4, 2, 0, 2, 1)$$

Then the discrete Haar wavelet transform of f can be computed as follows:

s \ t	0	1	2	3	4	5	6	7
$a_0 = f$	1	3	4	4	2	0	2	1
a_{-1}	2.83	5.66	1.41	2.12	–	–	–	–
c_{-1}	−1.41	0.00	1.41	0.71	–	–	–	–
a_{-2}	6	2.5	–	–	–	–	–	–
c_{-2}	−2	−0.5	–	–	–	–	–	–
a_{-3}	6.01	–	–	–	–	–	–	–
c_{-3}	2.47	–	–	–	–	–	–	–

$$\text{DWT}_{\text{haar}}(f) = (6.01, 2.47, -2, -0.5, -1.41, 0.00, 1.41, 0.71)$$

Haar Wavelets / Computing Coefficients / Example



Haar Wavelets / Computing Coefficients / Example

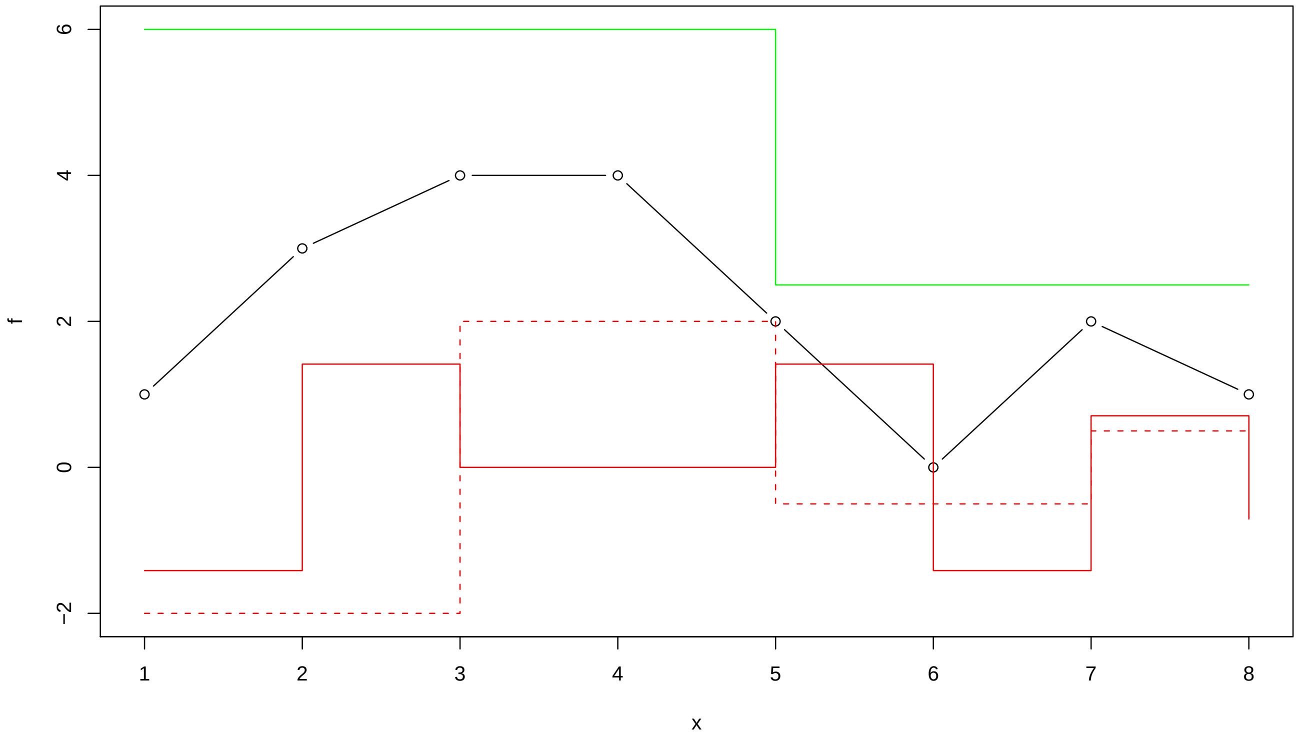
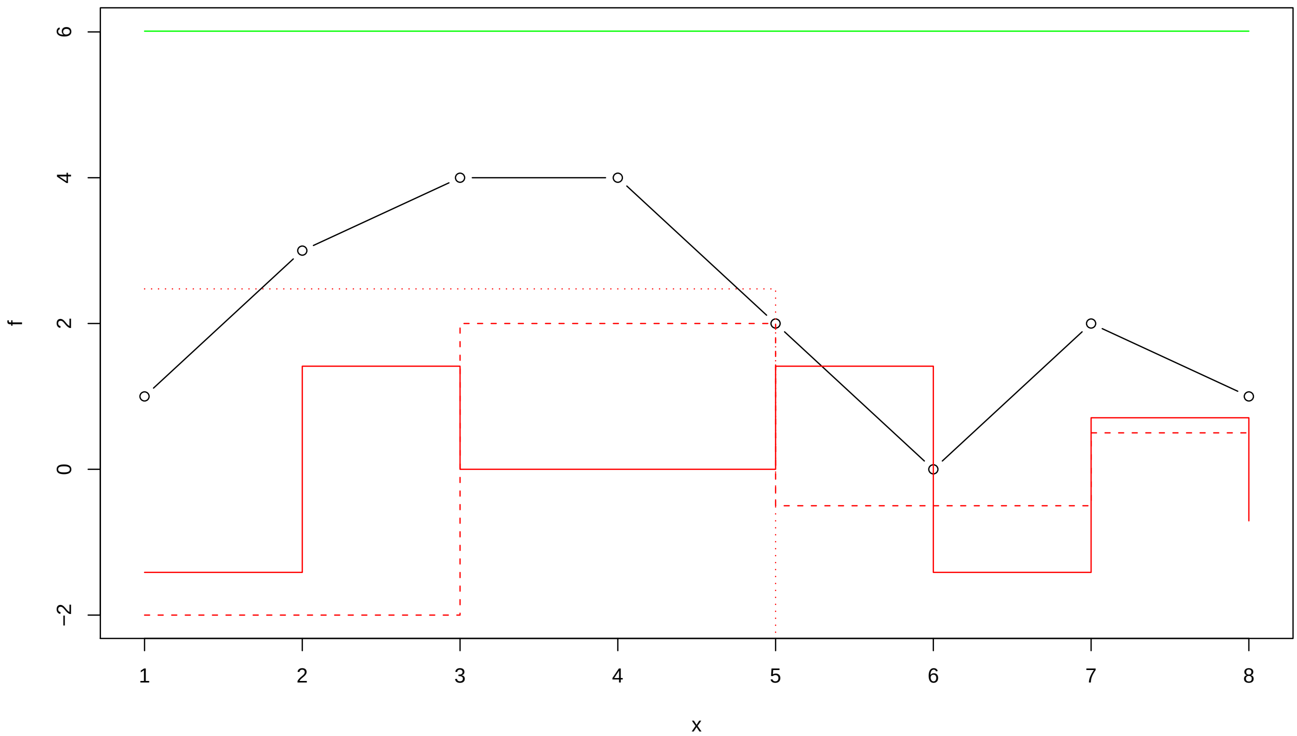


Image Analysis / 1. Haar Wavelets

Haar Wavelets / Computing Coefficients / Example



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Haar Wavelets / Computing Coefficients

dwt-haar(sequence $f = (f(x))_{x=0,\ldots,2^n-1}$) :
$c := (c_{s,t})_{s=0,\ldots,n-1;\ t=0,\ldots,2^s-1} := 0$
$a := (a_{s,t})_{s=0,\ldots,n;\ t=0,\ldots,2^s-1} := 0$
$a_{n,t} := f(t), \quad t = 0, \ldots, 2^n - 1$
for $s := n-1, \ldots, 0$ **do**
    for $t := 0, \ldots, 2^s - 1$ **do**
        $a_{s,t} := (a_{s+1,2t} + a_{s+1,2t+1})/\sqrt{2}$
        $c_{s,t} := (a_{s+1,2t} - a_{s+1,2t+1})/\sqrt{2}$
    od
od
return $(a_{0,0}, c)$

Haar Wavelets / Inverse Discrete Wavelet Transform

The DWT easily can be inverted: from

$$a_{s,t} = \frac{1}{\sqrt{2}}(a_{s+1,2t} + a_{s+1,2t+1})$$
$$c_{s,t} = \frac{1}{\sqrt{2}}(a_{s+1,2t} - a_{s+1,2t+1})$$

we get

$$a_{s+1,2t} = \sqrt{2}\,(a_{s,t} + c_{s,t})/2$$
$$a_{s+1,2t+1} = \sqrt{2}\,(a_{s,t} - c_{s,t})/2$$

Haar Wavelets / Inverse Discrete Wavelet Transform

idwt-haar(coefficients $c = (c_{s,t})_{s=0,\ldots,n-1;\ t=0,\ldots,2^s-1}, a')$:
$a := (a_{s,t})_{s=0,\ldots,n;\ t=0,\ldots,2^s-1} := 0$
$a_{0,0} := a'$
for $s := 0, \ldots, n-1$ **do**
    for $t := 0, \ldots, 2^s - 1$ **do**
        $a_{s+1,2t} := (a_{s,t} + c_{s,t})/\sqrt{2}$
        $a_{s+1,2t+1} := (a_{s,t} - c_{s,t})/\sqrt{2}$
    od
od
$f := (f(x))_{x=0,\ldots,2^n-1} := a_{n,x}, \quad x = 0, \ldots, 2^n - 1$
return f

Haar Wavelets / Matrix Notation

A single iteration from scale $s+1$ to s of the discrete Haar wavelet transform can be described by matrix multiplication:

$$\begin{pmatrix} a_0^s \\ c_0^s \\ a_1^s \\ c_1^s \\ \vdots \\ a_{n-1}^s \\ c_{n-1}^s \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & & & & & \\ 1 & -1 & & & & & & \\ & & 1 & 1 & & & & \\ & & 1 & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & 1 \\ & & & & & & 1 & -1 \end{pmatrix} \begin{pmatrix} a_0^{s+1} \\ a_1^{s+1} \\ a_2^{s+1} \\ \vdots \\ a_{n-1}^{s+1} \\ \vdots \\ a_{2n-1}^{s+1} \end{pmatrix}$$

Daubechies Wavelets / Definition

Ingrid Daubechies (*1954) generalized the Haar wavelets to a family of wavelets now called **Daubechies wavelets D_k**:

$$\begin{pmatrix} a_0^s \\ c_0^s \\ a_1^s \\ c_1^s \\ \vdots \\ a_{n-1}^s \\ c_{n-1}^s \end{pmatrix} = \begin{pmatrix} w_0 & w_1 & w_2 & \dots & w_{k-1} & & & & \\ w_{k-1} & -w_{k-2} & w_{k-3} & \dots & -w_0 & & & & \\ & & w_0 & w_1 & \dots & w_{k-1} & & & \\ & & w_{k-1} & -w_{k-2} & \dots & -w_0 & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ w_2 & \dots & w_{k-1} & & & & & w_0 & w_1 \\ w_{k-3} & \dots & -w_0 & & & & & w_{k-1} & -w_{k-2} \end{pmatrix} \begin{pmatrix} a_0^{s+1} \\ a_1^{s+1} \\ a_2^{s+1} \\ \vdots \\ a_{n-1}^{s+1} \\ \vdots \\ a_{2n-1}^{s+1} \end{pmatrix}$$

The coefficients $w_0, w_1, \dots, w_{k-1}$ are called the **wavelet filter coefficients**.

Daubechies Wavelets / Definition

The matrix D_k should satisfy two conditions:

1. **Orthogonality**, i.e., $D_k D_k^T = 1$:

$$\sum_{i=0}^{k-1} w_i^2 = 1$$

$$\sum_{i=0}^{k-1-2m} w_i\, w_{2m+i} = 0, \quad m = 1, 2, \ldots, k/2 - 1$$

2. **Approximation of order $k/2$**, i.e., the first $k/2$ moments vanish.

For D_4 this means:

$$w_3 - w_2 + w_1 - w_0 = 0$$
$$0w_3 - 1w_2 + 2w_1 - 3w_0 = 0$$

Daubechies Wavelets / Definition

In general, this are k conditions for the k coefficients of D_K leading to a unique solution:

$$w(D_2) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$
$$w(D_4) = (\frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}})$$

D_2 is the Haar wavelet.

$w(D_6)$ also can be computed analytically; the coefficients of the higher order Daubechies wavelets can only be computed numerically.

Daubechies Wavelets / DWT Algorithm

dwt-daubechies(sequence $f = (f(x))_{x=0,\ldots,2^n-1}, k)$:
$w := (w(x))_{x=0,\ldots,k-1} :=$ getDaubechiesWaveletCoefficients(k)
$c := (c_{s,t})_{s=0,\ldots,n-1;\, t=0,\ldots,2^s-1} := 0$
$a := (a_{s,t})_{s=0,\ldots,n;\, t=0,\ldots,2^s-1} := 0$
$a_{n,t} := f(t), \quad t = 0, \ldots, 2^n - 1$
for $s := n-1, \ldots, 0$ **do**
  for $t := 0, \ldots, 2^s - 1$ **do**
    $a_{s,t} := 0$
    $c_{s,t} := 0$
    for $x := 0, \ldots, k-1$ **do**
      $a_{s,t} := a_{s,t} + a_{s+1, 2t+x \bmod 2^{s+1}}\, w(x)$
      $c_{s,t} := c_{s,t} + a_{s+1, 2t+x \bmod 2^{s+1}}\, (-1)^x\, w(k-1-x)$
    od
  od
od
return $(a_{0,0}, c)$

Generic DWT Algorithm (1/2)

dwt-generic(sequence $f = (f(x))_{x=0,\ldots,2^n-1}$, wavelet transform $W)$:
$c := (c_{s,t})_{s=0,\ldots,n-1;\, t=0,\ldots,2^s-1} := 0$
$a := (a_{s,t})_{s=0,\ldots,n;\, t=0,\ldots,2^s-1} := 0$
$a_{n,t} := f(t), \quad t = 0, \ldots, 2^n - 1$
for $s := n-1, \ldots, 0$ **do**
  $(c_{s,.}, a_{s,.}) :=$ dwt-iteration$(a_{s+1,.}, W)$
od
return $(a_{0,0}, c)$

dwt-iteration(sequence $f = (f(x))_{x=0,\ldots,2^{n+1}-1}$, wavelet transform $W)$:
$a := (a_t)_{t=0,\ldots,2^n-1} := 0$
$c := (c_t)_{t=0,\ldots,2^n-1} := 0$
for $t := 0, \ldots, 2^n - 1$ **do**
  $(a_t, c_t) := W((f_{2t+x \bmod 2^{n+1}})_{x=0,\ldots,2^{n+1}-1})$
od
return (c, a)

Generic DWT Algorithm (2/2)

W-haar(sequence $a = (a(t))_{t=0,\ldots,2^n-1}$) :
return $((a_0 + a_1)/\sqrt{2}, (a_0 - a_1)/\sqrt{2})$

W-daubechies-k(sequence $a = (a(t))_{t=0,\ldots,2^n-1}$) :
$w := (w(x))_{x=0,\ldots,k-1} :=$ getDaubechiesWaveletCoefficients(k)
$a' := 0$
$c := 0$
for $x := 0, \ldots, k-1$ **do**
    $a' := a' + a_x\, w(x)$
    $c' := c' + a_x\, (-1)^x\, w(k-1-x)$
od
return (a', c')

Daubechies Wavelets / Inverse DWT Algorithm

As D_k is orthogonal, one can easily compute the inverse of the DWT via:

$$\begin{pmatrix} a_0^{s+1} \\ a_1^{s+1} \\ a_2^{s+1} \\ \vdots \\ a_{n-1}^{s+1} \\ \vdots \\ a_{2n-1}^{s+1} \end{pmatrix} = \begin{pmatrix} w_0 & w_1 & w_3 & \ldots & w_{k-1} & & & & \\ w_{k-1} & -w_{k-2} & w_3 & \ldots & -w_0 & & & & \\ & & w_0 & w_1 & \ldots & w_{k-1} & & & \\ & & w_{k-1} & -w_{k-2} & \ldots & -w_0 & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ w_2 & \ldots & w_{k-1} & & & & & w_0 & w_1 \\ w_2 & \ldots & -w_{k-1} & & & & & w_0 & -w_1 \end{pmatrix}^T \begin{pmatrix} a_0^s \\ c_0^s \\ a_1^s \\ c_1^s \\ \vdots \\ a_{n-1}^s \\ c_{n-1}^s \end{pmatrix}$$

Daubechies Wavelets / Inverse DWT Algorithm

idwt-daubechies(coefficients $c = (c_{s,t})_{s=0,\ldots,n-1;\ t=0,\ldots,2^s-1}, a', k)$:
$w := (w(x))_{x=0,\ldots,k-1} :=$ getDaubechiesWaveletCoefficients(k)
$a := (a_{s,t})_{s=0,\ldots,n;\ t=0,\ldots,2^s-1} := 0$
$a_{0,0} := a'$
for $s := 0, \ldots, n-1$ **do**
    for $t := 0, \ldots, 2^s - 1$ **do**
        for $x := 0, \ldots, k-1$ **do**
            $a_{s+1,2t+x \bmod 2^{s+1}} := a_{s+1,2t+x \bmod 2^{s+1}} + a_{s,t+x \bmod 2^s}\, w(x)$
            $a_{s+1,2t+1+x \bmod 2^{s+1}} := a_{s+1,2t+1+x \bmod 2^{s+1}} + c_{s,t+x \bmod 2^s}\, (-1)^x\, w(k-1-x)$
        od
    od
od
$f := (f(x))_{x=0,\ldots,2^n-1} := a_{n,x}, \quad x = 0, \ldots, 2^n - 1$
return f

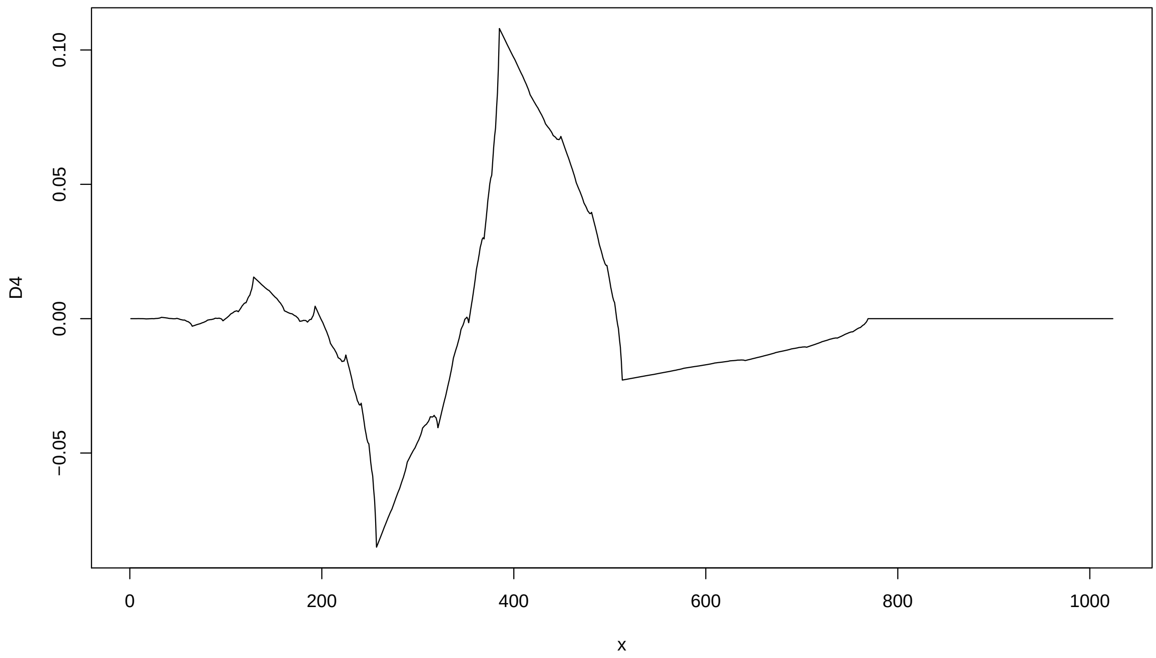
Daubechies Wavelets / Daubechies Wavelet Basis Functions

Daubechies wavelets have been defined implicitely by their wavelet coefficients in the DWT.

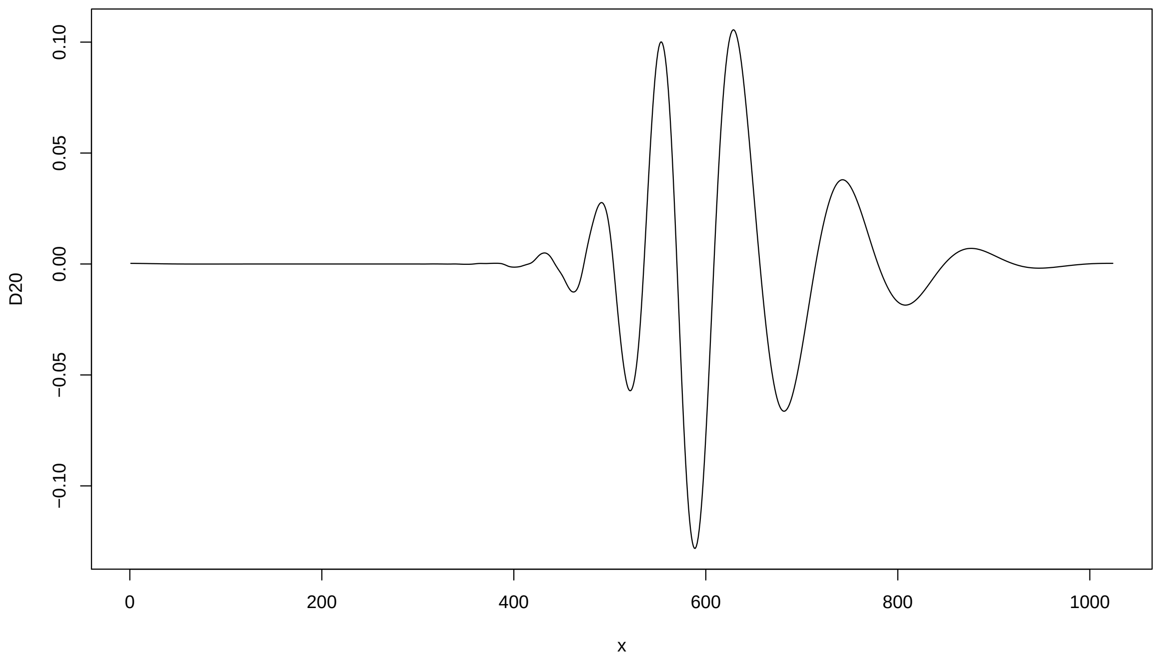
But how does a Daubechies wavelet look like?

We run a unit vector of length 1024 through IDWT, i.e., we set all but one coefficient to zero.

Daubechies Wavelets / D_4 Wavelet Basis Function ($s = 8, t = 3$)



Daubechies Wavelets / D_{20} Wavelet Basis Function ($s = 6, t = 3$)



1. Haar Wavelets
2. Daubechies Wavelets
3. **Two-dimensional Wavelets**

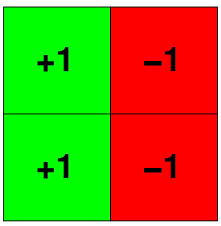
Two-dimensional Haar **Mother Wavelets**

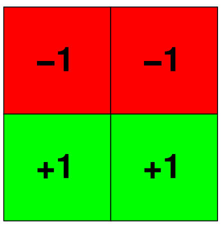
$$\text{haar}^b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad b \in \{1, 2, 3\}$$

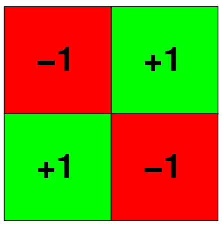
$$\text{haar}^1(x, y) := \begin{cases} +1, & \text{if } x \in [0, \frac{1}{2}) \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \\ 0, & \text{else} \end{cases}$$

$$\text{haar}^2(x, y) := \begin{cases} +1, & \text{if } y \in [0, \frac{1}{2}) \\ -1, & \text{if } y \in [\frac{1}{2}, 1) \\ 0, & \text{else} \end{cases}$$

$$\text{haar}^3(x, y) := \begin{cases} +1, & \text{if } (x, y) \in [0, \frac{1}{2})^2 \cup [\frac{1}{2}, 1)^2 \\ -1, & \text{if } (x, y) \in [0, \frac{1}{2}) \times [\frac{1}{2}, 1 \cup [\frac{1}{2}, 1) \times [0, \frac{1}{2} \\ 0, & \text{else} \end{cases}$$



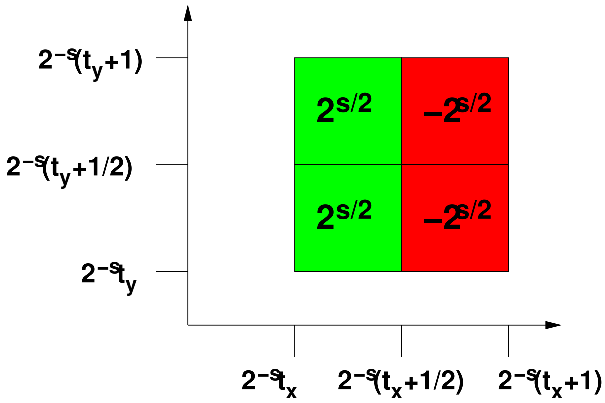




Two-dimensional Haar Basis Functions

The scaled and translated mother wavelets form a family of two-dimensional Haar basis functions:

$$\psi^b_{s,t_x,t_y}(x) := 2^s \cdot \text{haar}^b(2^s x - t_x, 2^s y - t_y), \quad b \in \{1,2,3\}$$



Orthogonality of Haar Basis Functions

Obviously, two distinct Haar basis functions

$$\psi^b_{s,t_x,t_y} \quad \text{and} \quad \psi^{b'}_{s',t'_x,t'_y}$$

with $s, t_x, t_y, s', t'_x, t'_y \in \mathbb{Z}$, $b, b' \in \{1,2,3\}$ are orthogonal:

$$\langle \psi^b_{s,t_x,t_y}, \psi^{b'}_{s',t'_x,t'_y} \rangle := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^b_{s,t_x,t_y}(x) \cdot \psi^{b'}_{s',t'_x,t'_y}(x)\, dx dy = 0$$

And

$$\langle \psi^b_{s,t_x,t_y}, \psi^b_{s,t_x,t_y} \rangle = 1$$

Proof. Analogously to the one-dimensional case.

2D Haar Wavelets / Discrete Wavelet Transform

A finite discrete signal f of size $2^n \times 2^n$ can be represented by a finite sum of 2-dimensional Haar wavelets:

$$f(x) = a_{-n,0,0} + \sum_{b=1}^{3} \sum_{s=-n}^{-1} \sum_{t_x=0}^{2^{n+s}-1} \sum_{t_y=0}^{2^{n+s}-1} c^b_{s,t_x,t_y} \cdot 2^s \, \text{haar}^b(2^s\, x - t_x, 2^s\, y - t_y)$$

The initial a values are just the signal values:

$$\begin{aligned} a_{s=0,t_x,t_y} &:= \int_{2^{-s}\,t_x}^{2^{-s}\,(t_x+1)} \int_{2^{-s}\,t_y}^{2^{-s}\,(t_y+1)} f(x,y)\,dxdy \\ &= \int_{t_x}^{t_x+1} \int_{t_y}^{t_y+1} f(x,y)\,dxdy \\ &= \sum_{x=t_x}^{<t_x+1} \sum_{y=t_y}^{<t_y+1} f(x,y) = f(t_x,t_y) \end{aligned}$$

2D Haar Wavelets / Computing Coefficients

The values of integrals with a simple rectangle impulse on different scales can be computed recursively:

$$\begin{aligned} a_{s,t_x,t_y} &:= 2^s \int_{2^{-s}\,t_x}^{2^{-s}\,(t_x+1)} \int_{2^{-s}\,t_y}^{2^{-s}\,(t_y+1)} f(x,y)\,dxdy \\ a_{s,t_x,t_y} &= \frac{1}{2}\left(a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y} + a_{s+1,2t_x,2t_y+1} + a_{s+1,2t_x+1,2t_y+1}\right) \end{aligned}$$

The coefficients of the Haar wavelet can be computed from these values via

$$\begin{aligned} c^1_{s,t_x,t_y} &= \frac{1}{2}\left(a_{s+1,2t_x,2t_y} + a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y} - a_{s+1,2t_x+1,2t_y+1}\right) \\ c^2_{s,t_x,t_y} &= \frac{1}{2}\left(a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y} - a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y+1}\right) \\ c^1_{s,t_x,t_y} &= \frac{1}{2}\left(a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y+1} - a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y}\right) \end{aligned}$$

Haar Wavelets / Computing Coefficients

dwt2d-haar(image $f = (f(x,y))_{x=0,\ldots,2^n-1, y=0,\ldots,2^n-1})$:

$c := (c^b_{s,t_x,t_y})_{s=0,\ldots,n-1;\ t_x=0,\ldots,2^s-1;\ t_y=0,\ldots,2^s-1;\ b\in\{1,2,3\}} := 0$

$a := (a_{s,t_x,t_y})_{s=0,\ldots,n;\ t_x=0,\ldots,2^s-1;\ t_y=0,\ldots,2^s-1} := 0$

$a_{n,t_x,t_y} := f(t_x,t_y), \quad t_x = 0,\ldots,2^n-1, t_y = 0,\ldots,2^n-1$

for $s := n-1,\ldots,0$ **do**

  for $t_x := 0,\ldots,2^s-1$ **do**

    for $t_y := 0,\ldots,2^s-1$ **do**

$$a_{s,t_x,t_y} := (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y} + a_{s+1,2t_x,2t_y+1} + a_{s+1,2t_x+1,2t_y+1})/2$$

$$c^1_{s,t_x,t_y} := (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y} - a_{s+1,2t_x+1,2t_y+1})/2$$

$$c^2_{s,t_x,t_y} := (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y} - a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y+1})/2$$

$$c^3_{s,t_x,t_y} := (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y+1} - a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y})/2$$

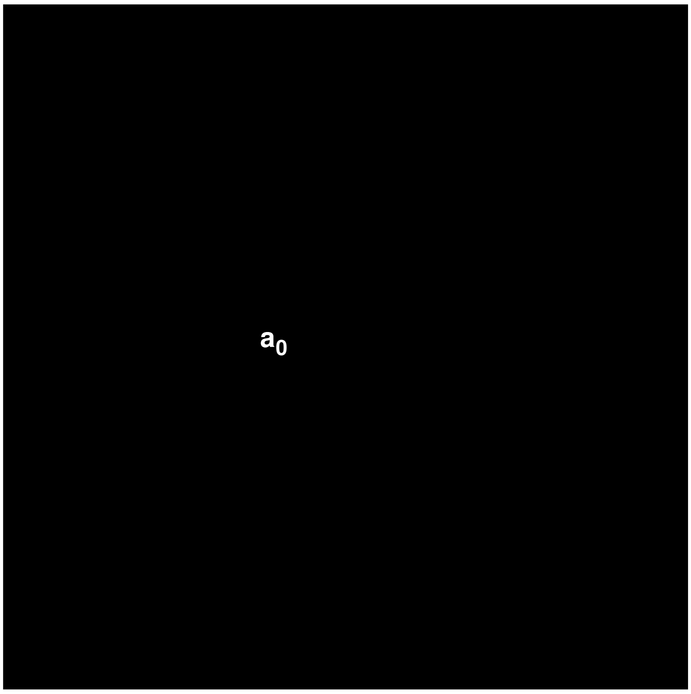
    od

  od

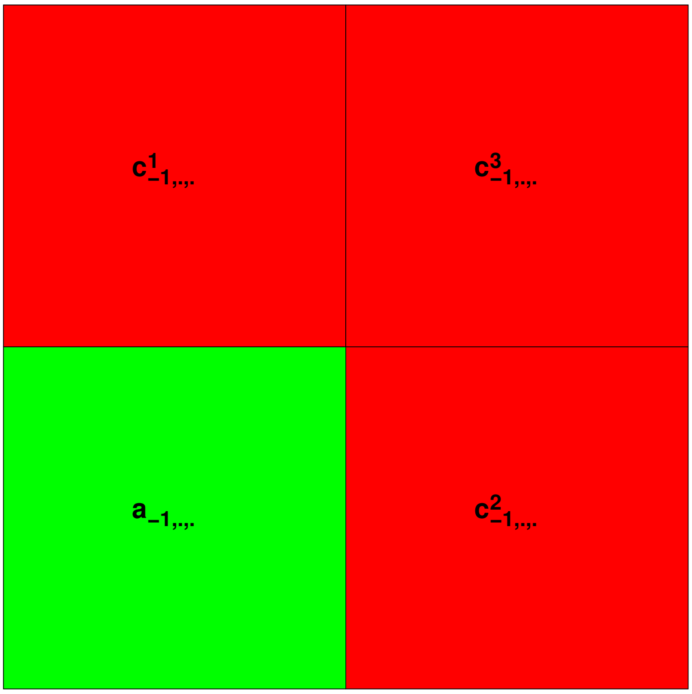
od

return $(a_{0,0,0}, c)$

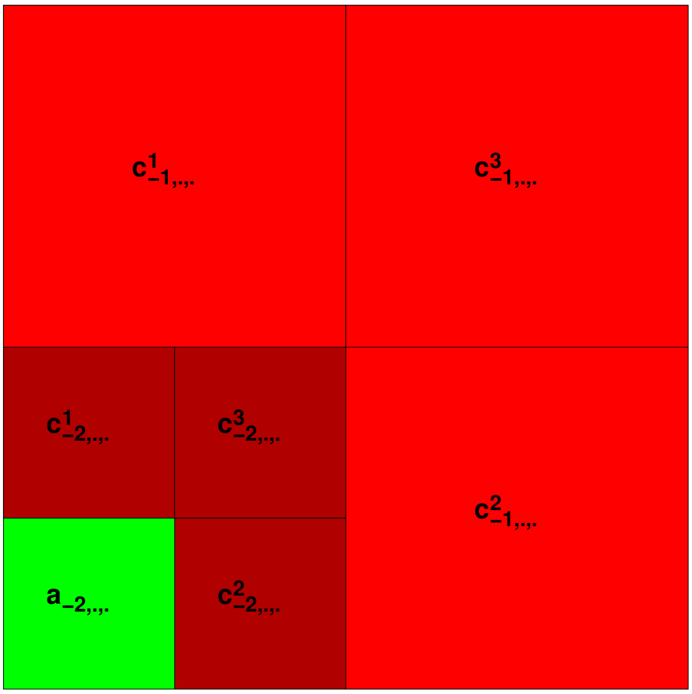
Displaying 2D DWTs



Displaying 2D DWTs



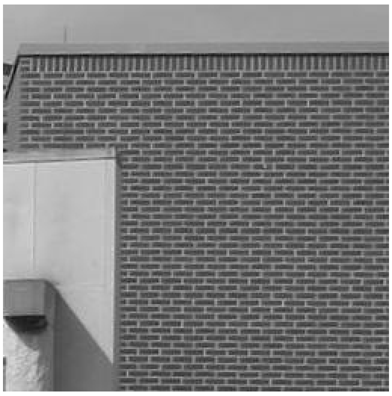
Displaying 2D DWTs

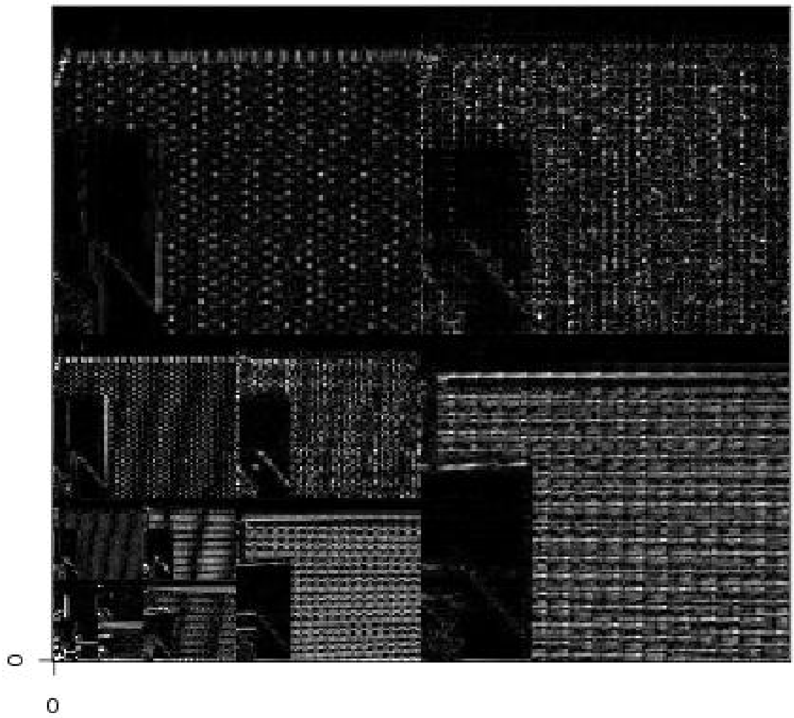


Example

Example

Another Example





Separable 2D Wavelets Bases / Scaling Function

Many 2D wavelet bases can be constructed from 1D wavelet bases and a suitable **scaling function** ϕ (also called **father wavelet**).

For the Haar wavelets the scaling function is just the rectangle impulse:

$$\phi(x) := \begin{cases} 1, & \text{if } x \in [0,1) \\ 0, & \text{else} \end{cases}$$

In the same manner as for the wavelet functions, one defines scaled and translated variants:

$$\phi_{s,t}(x) := 2^s \, \phi(2^s x - t)$$

Separable 2D Wavelets Bases / 2D Haar Basis

Obviously, the Haar basis wavelets can be constructed via

$$\psi^1(x,y) = \psi(x)\,\phi(y)$$
$$\psi^2(x,y) = \phi(x)\,\psi(y)$$
$$\psi^3(x,y) = \psi(x)\,\psi(y)$$

and

$$\phi(x,y) = \phi(x)\,\phi(y)$$

is a suitable 2D scaling function.

Separable wavelet bases allow a generic DWT that

1. applies a 1D DWT to each row of the image and then
2. applies another 1D DWT to each column of the result.

Generic DWT Algorithm

dwt2d-generic(image $f = (f(x,y))_{x=0,\ldots,2^n-1, y=0,\ldots,2^n-1}$, wavelet transform W) :
$c := (c^b_{s,t_x,t_y})_{s=0,\ldots,n-1;\ t_x=0,\ldots,2^s-1;\ t_y=0,\ldots,2^s-1;\ b\in\{1,2,3\}} := 0$
$a := (a_{s,t_x,t_y})_{s=0,\ldots,n;\ t_x=0,\ldots,2^s-1;\ t_y=0,\ldots,2^s-1} := 0$
$a_{n,t_x,t_y} := f(t_x,t_y), \quad t_x = 0,\ldots,2^n-1, t_y = 0,\ldots,2^n-1$
for $s := n-1,\ldots,0$ **do**
    $a' := (a'_{t_x,t_y})_{t_x=0,\ldots,2^{s+1}-1;\ t_y=0,\ldots,2^s-1} := 0$
    $c' := (c'_{t_x,t_y})_{t_x=0,\ldots,2^{s+1}-1;\ t_y=0,\ldots,2^s-1} := 0$
    for $t_x := 0,\ldots,2^{s+1}-1$ **do**
        $(a'_{t_x,.}, c'_{t_x,.}) :=$ dwt-iteration$(a_{s+1,t_x,.}, W)$
    od
    for $t_y := 0,\ldots,2^s-1$ **do**
        $(a_{s,.,t_y}, c^2_{s,.,t_y}) :=$ dwt-iteration$(a'_{.,t_y}, W)$
        $(c^1_{s,.,t_y}, c^3_{s,.,t_y}) :=$ dwt-iteration$(c'_{.,t_y}, W)$
    od
od
return $(a_{0,0,0}, c)$