Image Analysis

3. Fourier Transform

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Image Analysis

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Course on Image Analysis, winter term 2011/12 1/65

Periodic Functions

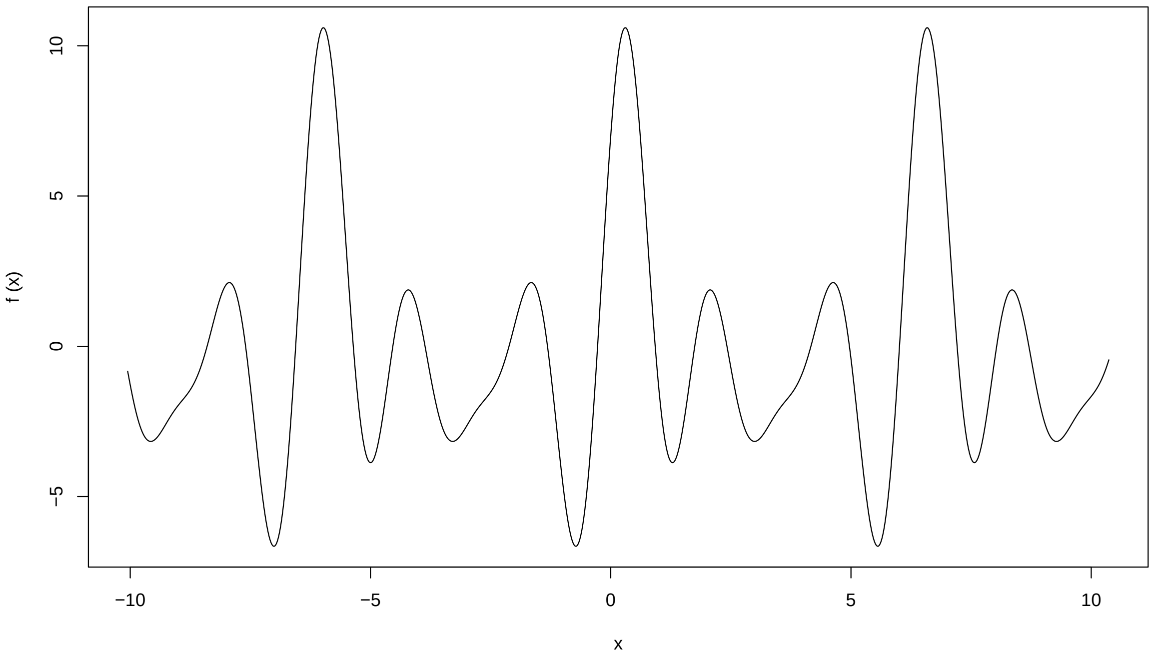
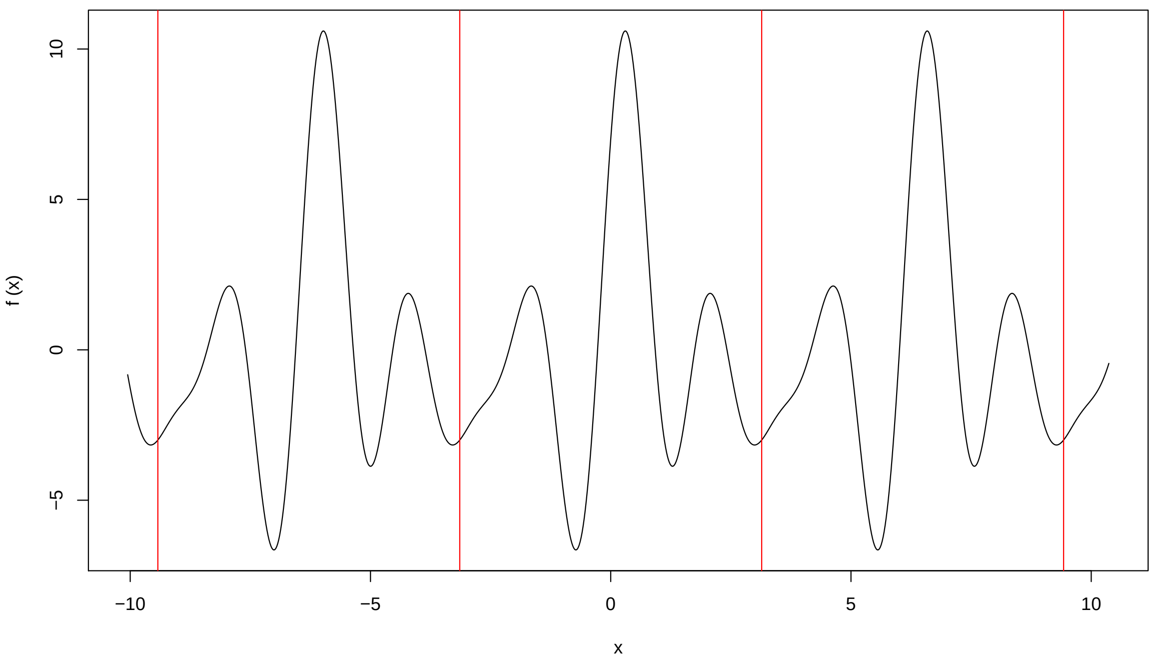


Image Analysis / 1. Fourier Series Representation

Periodic Functions



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Periodic Functions

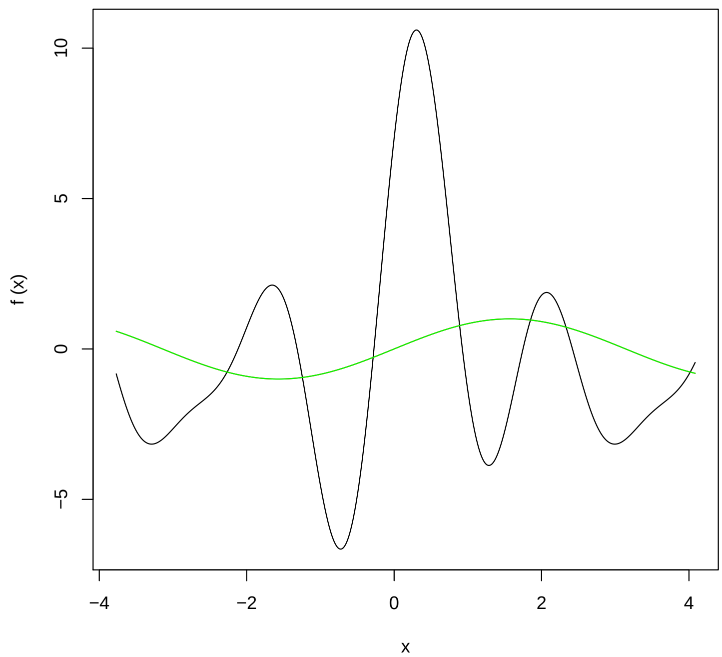
A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is called **T-periodic** if

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}$$

Example: the functions $\sin$ and $\cos$ are 2π-periodic.

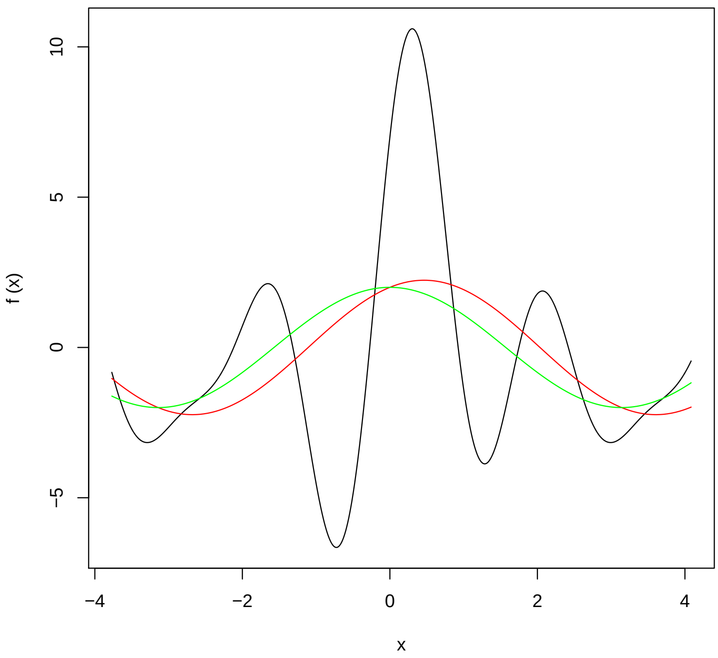
Example: the function $\sin(\omega x)$ is $2\pi/\omega$-periodic.

Periodic Functions / Approximation



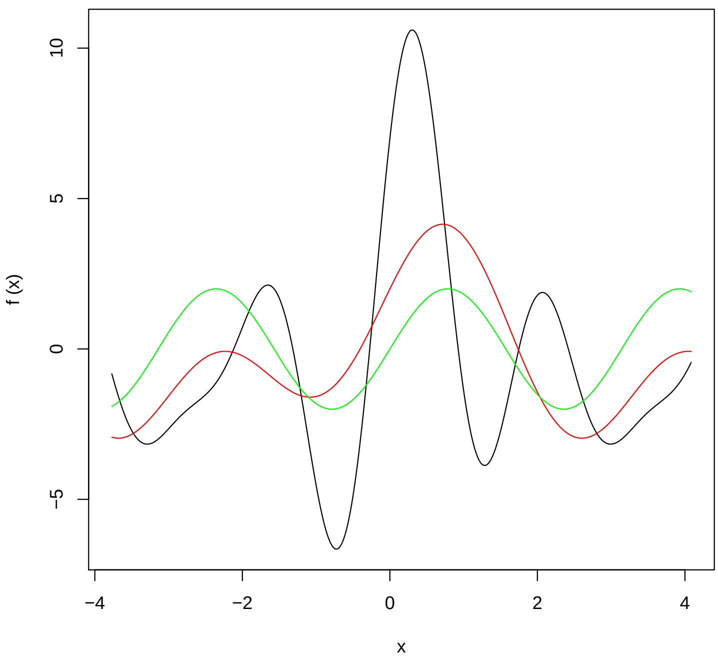
$$\hat{f}(x) = 1\sin(x)$$

Periodic Functions / Approximation



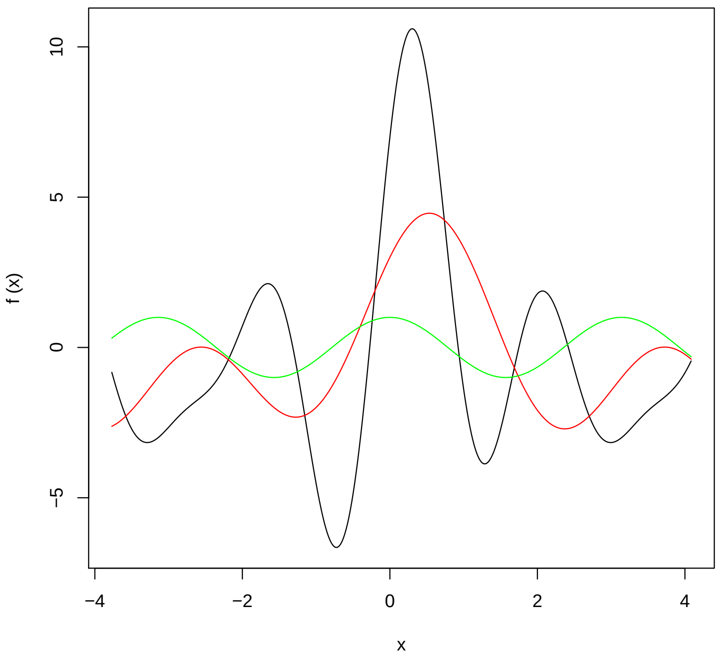
$$\hat{f}(x) = 1\sin(x) + 2\cos x$$

Periodic Functions / Approximation



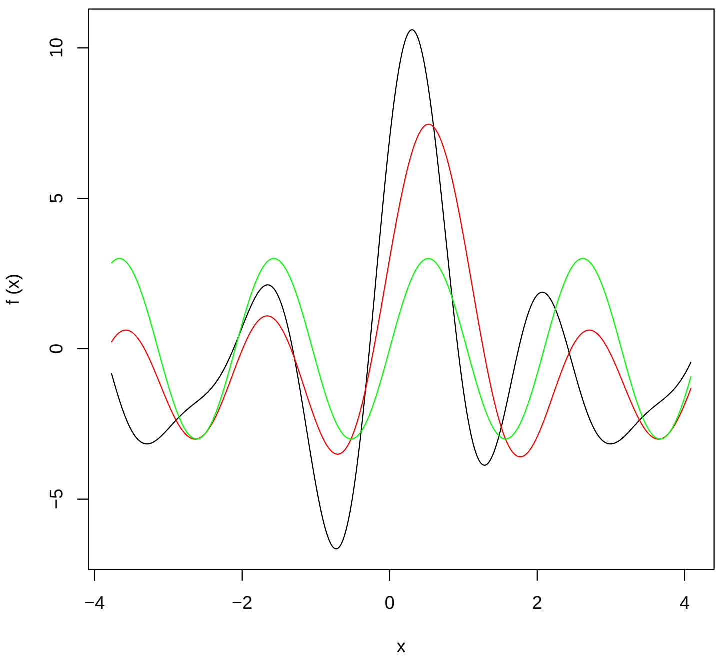
$$\hat{f}(x) = 1\sin(x) + 2\cos x + 2\sin 2x$$

Periodic Functions / Approximation



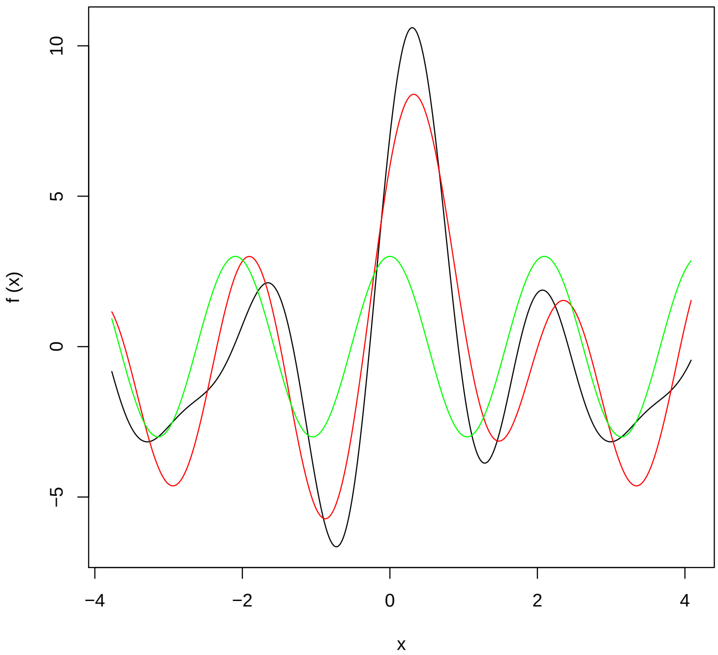
$$\hat{f}(x) = 1\sin(x) + 2\cos x + 2\sin 2x + 1\cos 2x$$

Periodic Functions / Approximation



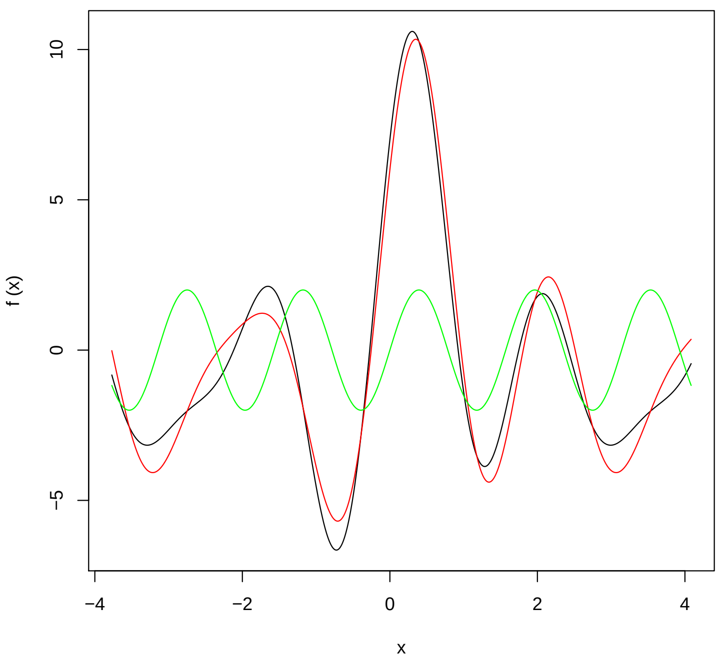
$$\hat{f}(x) = 1\sin(x) + 2\cos x + 2\sin 2x + 1\cos 2x + 3\sin 3x$$

Periodic Functions / Approximation



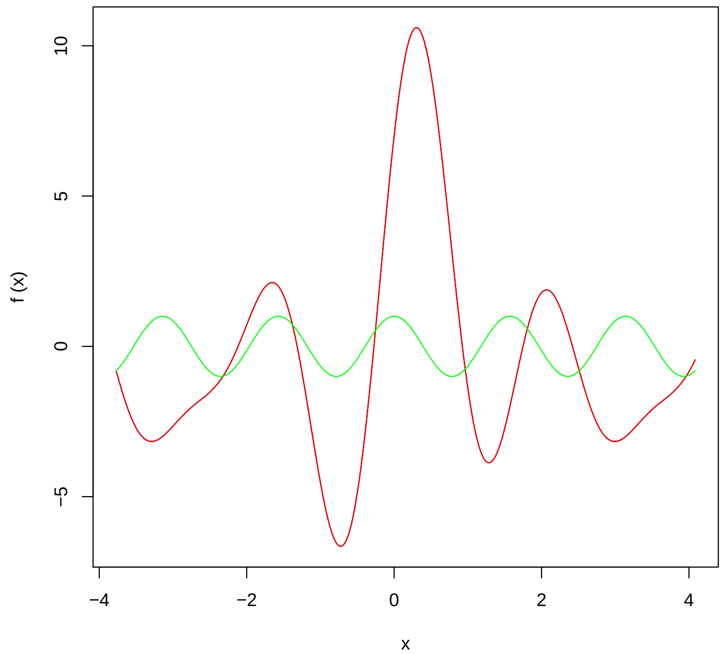
$$\textcolor{red}{\hat{f}(x)} = 1\sin(x) + 2\cos x + 2\sin 2x + 1\cos 2x + 3\sin 3x + \textcolor{green}{3\cos 3x}$$

Periodic Functions / Approximation



$$\textcolor{red}{\hat{f}(x)} = 1\sin(x) + 2\cos x + 2\sin 2x + 1\cos 2x + 3\sin 3x + 3\cos 3x + \textcolor{green}{2\sin 4x}$$

Periodic Functions / Approximation



$$\hat{f}(x) = 1\sin(x) + 2\cos x + 2\sin 2x + 1\cos 2x + 3\sin 3x + 3\cos 3x + 2\sin 4x + 1\cos 4x$$

Fourier Series Representation

Theorem (Fourier Series Representation). Any continuous, differentiable and T-periodic function $f : \mathbb{R} \to \mathbb{C}$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega x + b_k \sin k\omega x, \quad \omega := \frac{2\pi}{T}$$

with coefficients $a_k, b_k \in \mathbb{C}$, called a **Fourier Series of f**.

How to compute Fourier coefficients a_k, b_k for a given function f ?

Jean Baptiste Joseph Fourier (1768–1830),
French Mathematician and Physicist

Note. Actually, the class of functions that can be represented as Fourier series is much larger (see, e.g., [**?**, p. 314]).

Trigonometric Addition Formulas

Lemma (Trigonometric Addition Formulas). For all $x, y \in \mathbb{R}$:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

Some Trigonometric Integrals

$$\int \cos ax \cos bx \, dx = \begin{cases} \frac{1}{2}\left(\frac{\sin(a+b)x}{a+b} + \frac{\sin(a-b)x}{a-b}\right), & \text{if } a \neq b \\ \frac{\sin(2ax)+2ax}{4a}, & \text{else} \end{cases}$$

$$\int \sin ax \sin bx \, dx = \begin{cases} -\frac{1}{2}\left(\frac{\sin(a+b)x}{a+b} - \frac{\sin(a-b)x}{a-b}\right), & \text{if } a \neq b \\ -\frac{\sin(2ax)-2ax}{4a}, & \text{else} \end{cases}$$

$$\int \sin ax \cos bx \, dx = \begin{cases} -\frac{1}{2}\left(\frac{\cos(a+b)x}{a+b} + \frac{\cos(a-b)x}{a-b}\right), & \text{if } a \neq b \\ \frac{\sin^2(ax)}{2a}, & \text{else} \end{cases}$$

Some Trigonometric Integrals

The six formulas easily can be proven by differentiation, e.g.,

$$\int \cos ax \cos bx \, dx \stackrel{?}{=} \frac{1}{2}\left(\frac{\sin(a+b)x}{a+b} + \frac{\sin(a-b)x}{a-b}\right)$$

for $a \neq b$. Derivation $\frac{d}{dx}$ yields:

$$\begin{aligned}
\cos ax \cos bx &\stackrel{?}{=} \frac{1}{2}\left(\frac{(a+b)\cos(a+b)x}{a+b} + \frac{(a-b)\cos(a-b)x}{a-b}\right) \\
&= \frac{1}{2}(\cos(a+b)x + \cos(a-b)x) \\
&= \frac{1}{2}(\cos ax \cos bx - \sin ax \sin bx + \cos ax \cos(-bx) - \sin ax \sin(-bx)) \\
&= \frac{1}{2}(\cos ax \cos bx - \sin ax \sin bx + \cos ax \cos bx + \sin ax \sin bx) \\
&= \cos ax \cos bx
\end{aligned}$$

Trigonometric Orthogonality Relations

Let $\omega \in \mathbb{R}^+$. The functions

$$\{\sin k\omega x \mid k \in \mathbb{N}, k > 0\} \cup \{\cos k\omega x \mid k \in \mathbb{N}, k > 0\}$$

are pairwise othogonal with respect to

$$\langle f, g \rangle := \int_{-\pi/\omega}^{+\pi/\omega} f(x)g(x)dx$$

i.e., for any two distinct such functions f, g

$$\langle f, g \rangle = \int_{-\pi/\omega}^{+\pi/\omega} f(x)g(x)dx = 0$$

but

$$\langle f, f \rangle = \frac{\pi}{\omega} \neq 0$$

Trigonometric Orthogonality Relations

Proof: let $f = \cos k\omega x$ and $g = \cos l\omega x$ with $k \neq l$, then:

$$\langle f, g\rangle = \int_{-\pi/\omega}^{+\pi/\omega} \cos k\omega x \cos l\omega x \, dx = \left[\frac{1}{2}\left(\frac{\sin(k+l)\omega x}{(k+l)\omega} + \frac{\sin(k-l)\omega x}{(k-l)\omega}\right)\right]_{-\pi/\omega}^{+\pi/\omega}$$
$$= \frac{1}{2}\left(\frac{\sin(k+l)\pi}{(k+l)\omega} + \frac{\sin(k-l)\pi}{(k-l)\omega} - \frac{\sin -(k+l)\pi}{(k+l)\omega} - \frac{\sin -(k-l)\pi}{(k-l)\omega}\right)$$
$$= \frac{\sin(k+l)\pi}{(k+l)\omega} + \frac{\sin(k-l)\pi}{(k-l)\omega} = 0$$

but

$$\langle f, f\rangle = \left[\frac{\sin(2k\omega x) + 2k\omega x}{4k\omega}\right]_{-\pi/\omega}^{+\pi/\omega} = \frac{\sin(2k\pi) + 2k\pi}{4k\omega} - \frac{\sin(-2k\pi) + (-2k\pi)}{4k\omega} = \frac{\pi}{\omega}$$

Fourier Series Representation

Theorem (Fourier Series Representation). Any continuous, differentiable and T-periodic function $f : \mathbb{R} \to \mathbb{C}$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega x + b_k \sin k\omega x, \quad \omega := \frac{2\pi}{T}$$

with coefficients $a_k, b_k \in \mathbb{C}$, called a **Fourier Series of f**.

How to compute Fourier coefficients a_k, b_k for a given function f ?

$$a_k = \frac{1}{\pi/\omega} \int_{-\pi/\omega}^{+\pi/\omega} f(x) \cos k\omega x \, dx$$
$$b_k = \frac{1}{\pi/\omega} \int_{-\pi/\omega}^{+\pi/\omega} f(x) \sin k\omega x \, dx$$

Note. Actually, the class of functions that can be represented as Fourier series is much larger (see, e.g., [**?**, p. 314]).

Fourier Series Representation

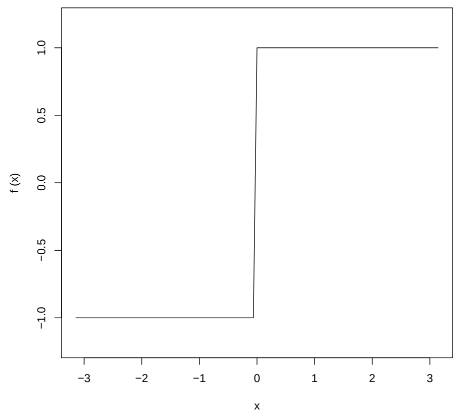
Proof.

$$
\begin{aligned}
a_k \overset{?}{=} & \frac{1}{\pi/\omega} \int_{-\pi/\omega}^{+\pi/\omega} f(x) \cos k\omega x \, dx \\
= & \frac{1}{\pi/\omega} \int_{-\pi/\omega}^{+\pi/\omega} \left(\frac{a_0}{2} + \sum_{l=1}^{\infty} a_l \cos l\omega x + b_l \sin l\omega x \right) \cos k\omega x \, dx \\
= & \frac{1}{\pi/\omega} \left(\int_{-\pi/\omega}^{+\pi/\omega} \frac{a_0}{2} \cos k\omega x \, dx + \sum_{l=1}^{\infty} \int_{-\pi/\omega}^{+\pi/\omega} a_l \cos l\omega x \cos k\omega x \, dx \right. \\
& \left. + \int_{-\pi/\omega}^{+\pi/\omega} b_l \sin l\omega x \cos k\omega x \, dx \right) \\
= & \frac{1}{\pi/\omega} \frac{\pi}{\omega} a_k \\
= & a_k
\end{aligned}
$$

Fourier Representation / Rectangular function

Let f be the 2π-periodic rectangular function

$$
f(x) = \begin{cases} -1, & \text{if } x \in (-\pi, 0) \\ 0, & \text{if } x \in \{-\pi, 0, \pi\} \\ +1, & \text{if } x \in (0, \pi) \end{cases}
$$



The Fourier representation of f is

$$
f(x) = \frac{4}{\pi} \sum_{k \in \mathbb{N} \text{ odd}} \frac{\sin kx}{k}
$$

Proof:

$$
b_k = \frac{2}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx \, dx = \frac{2}{\pi} 2 \int_{0}^{+\pi} \sin kx \, dx = \frac{4}{\pi} \left[-\frac{1}{k} \cos kx \right]_0^{\pi} = \begin{cases} -\frac{4}{\pi k}(0-1) = \frac{4}{\pi k}, & \text{if } k \text{ odd} \\ -\frac{4}{\pi k}(1-1) = 0, & \text{if } k \text{ even} \end{cases}
$$

$$
a_k = \frac{2}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx \, dx = 0
$$

$\rightsquigarrow$ in general, Fourier representations are infinite as in this example!

Fourier Representation / Rectangular function

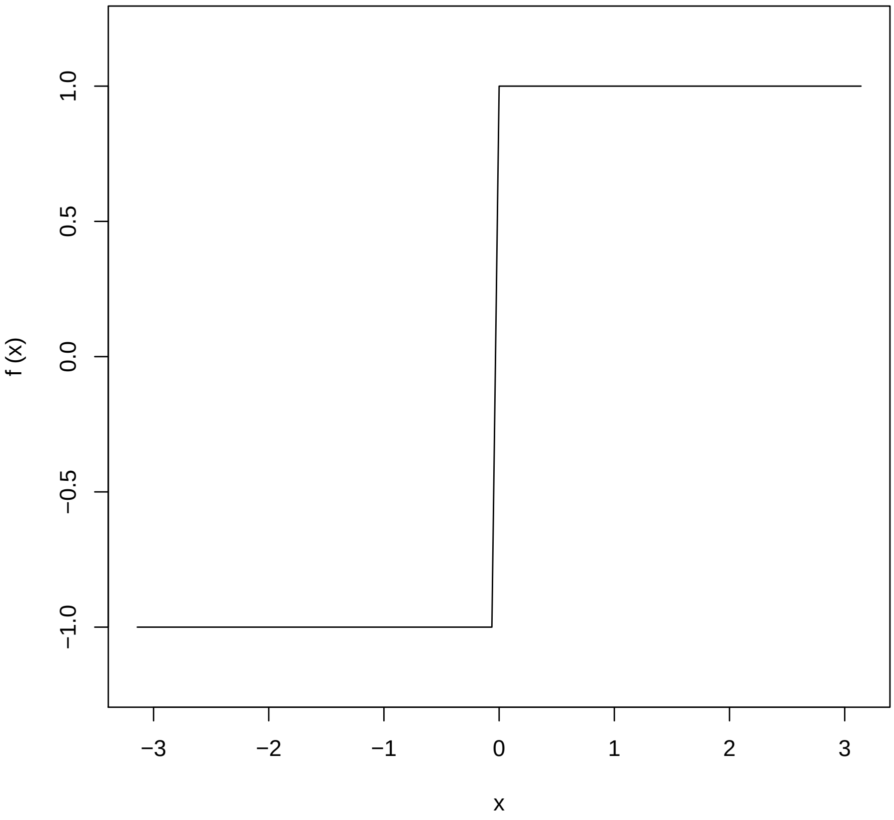
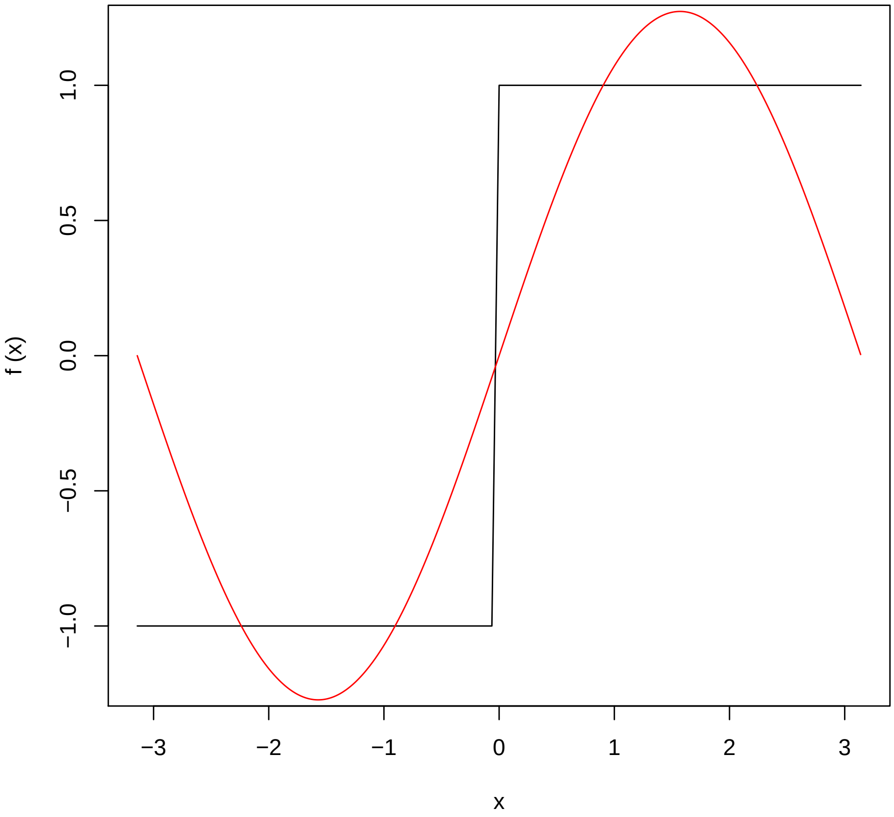


Image Analysis / 1. Fourier Series Representation

Fourier Representation / Rectangular function / $k = 1$



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Fourier Representation / Rectangular function / $k = 5$

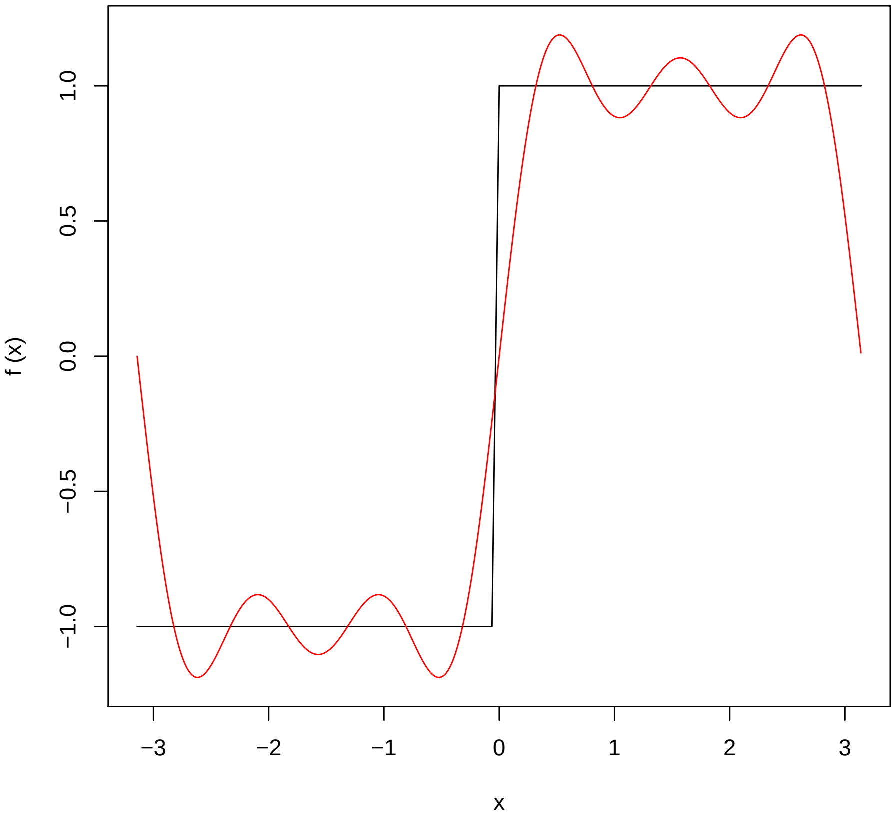
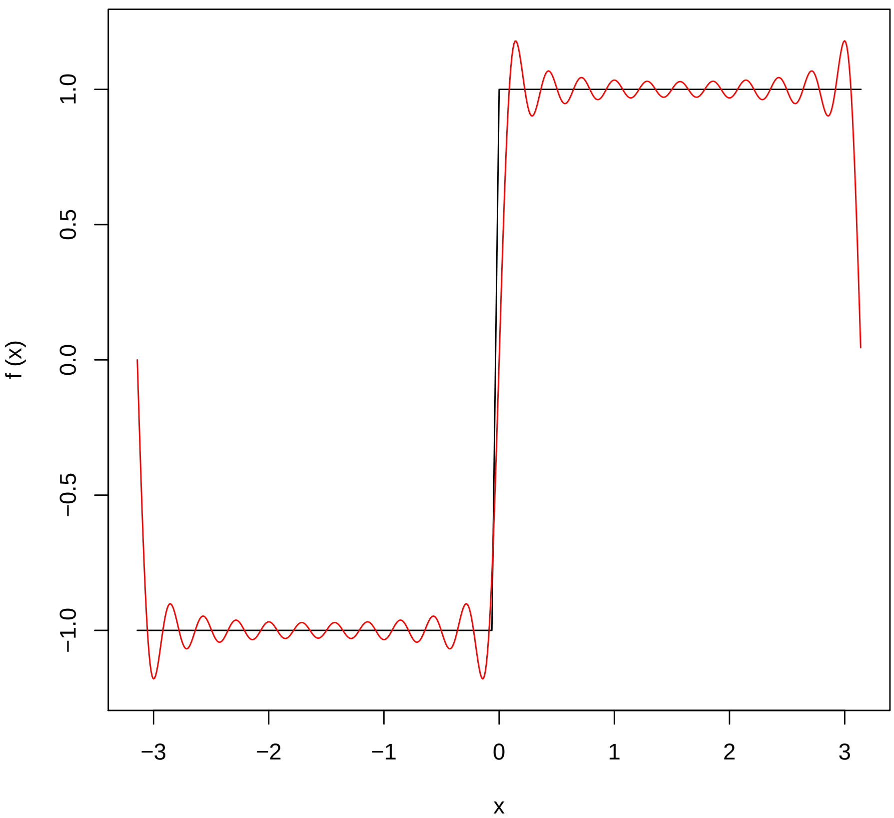


Image Analysis / 1. Fourier Series Representation

Fourier Representation / Rectangular function / $k = 21$



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Eulers Formula

$$\cos x := \sum_{n\in\mathbb{N}} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x := \sum_{n\in\mathbb{N}} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\exp x := \sum_{n\in\mathbb{N}} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Lemma (**Eulers formula**). For $x \in \mathbb{C}$:

$$e^{ix} = \cos(x) + i \cdot \sin(x), \quad \text{with } i := \sqrt{-1} \text{ the imaginary unit}$$

Proof:

$$\begin{aligned} e^{ix} &= \sum_{n\in\mathbb{N}} \frac{(ix)^n}{n!} \\ &= \sum_{n\in\mathbb{N}} \frac{(ix)^{2n}}{(2n)!} + \sum_{n\in\mathbb{N}} \frac{(ix)^{2n+1}}{(2n+1)!} \\ &= \sum_{n\in\mathbb{N}} (-1)^n \frac{x^{2n}}{(2n)!} + i \cdot \sum_{n\in\mathbb{N}} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \cos(x) + i \sin(x) \end{aligned}$$

Complex Fourier Series Representation

Theorem (Complex Fourier Series Representation). Any continuous, differentiable and T-periodic function $f : \mathbb{R} \to \mathbb{C}$ can be written as

$$f = \sum_{k\in\mathbb{Z}} c_k \, e^{ik\omega x}, \quad \omega := \frac{2\pi}{T}$$

with coefficients $c_k \in \mathbb{C}$, called a **Fourier Series of f**.

The coefficients of the complex Fourier series can be computed via

$$c_k = \frac{1}{2\pi/\omega} \int_{-\pi/\omega}^{+\pi/\omega} f(x) \, e^{-ik\omega x} dx$$

Complex Fourier Series Representation

Proof.

$$
\begin{aligned}
f &= \sum_{k\in\mathbb{Z}} c_k\, e^{ik\omega x} \\
&= \sum_{k\in\mathbb{Z}} c_k \,(\cos k\omega x + i \sin k\omega x) \\
&= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k})\ \cos k\omega x + (c_k - c_{-k}) i \sin k\omega x \\
&= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega x + b_k \sin k\omega x
\end{aligned}
$$

with

$$a_0 = 2c_0, \quad a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k})$$

and vice versa via:

$$c_0 = \frac{a_0}{2}, \quad c_k = \frac{1}{2}(a_k - ib_k), \quad c_{-k} = \frac{1}{2}(a_k + ib_k)$$

Fourier Transform

Let $f : \mathbb{R} \to \mathbb{C}$ be a function (that satisfies some regularity conditions). Then

$$\begin{aligned} F : \mathbb{R} &\to \mathbb{C} \\ \omega &\mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx \end{aligned}$$

exists for each ω and is a continuous function called **Fourier Transform of f** (aka **Fourier spectrum of f**).

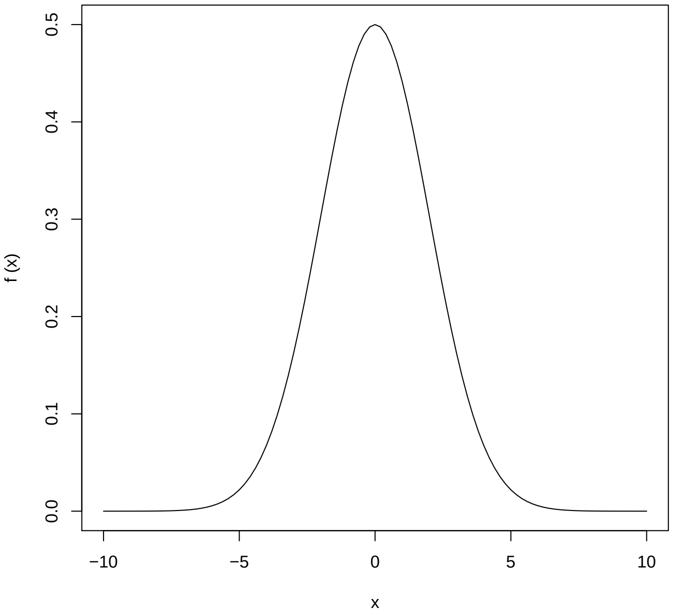
One can show that (if F also satisfies some regularity conditions):

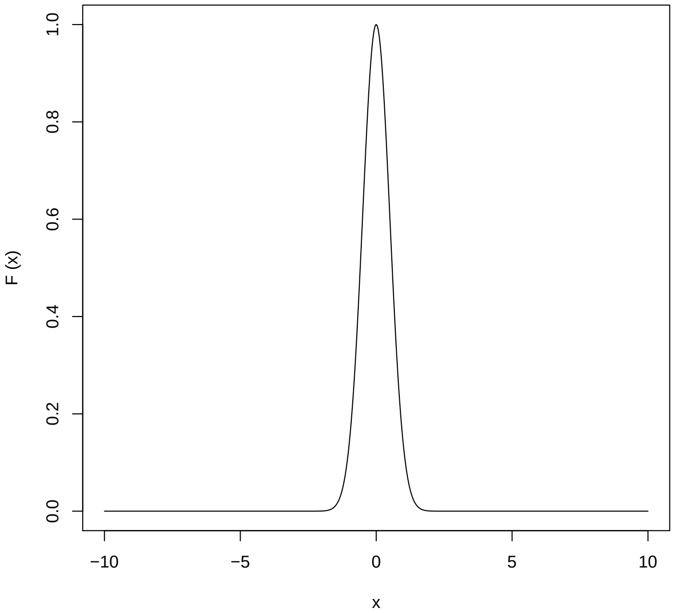
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} d\omega$$

This is called **Inverse Fourier Transform**.

We will write $\mathcal{F}(f) := F$ for the Fourier transform of a function f and $\mathcal{F}^{-1}(F)$ for the inverse Fourier transform of a function F.

Fourier Transforms / Examples / Gaussian

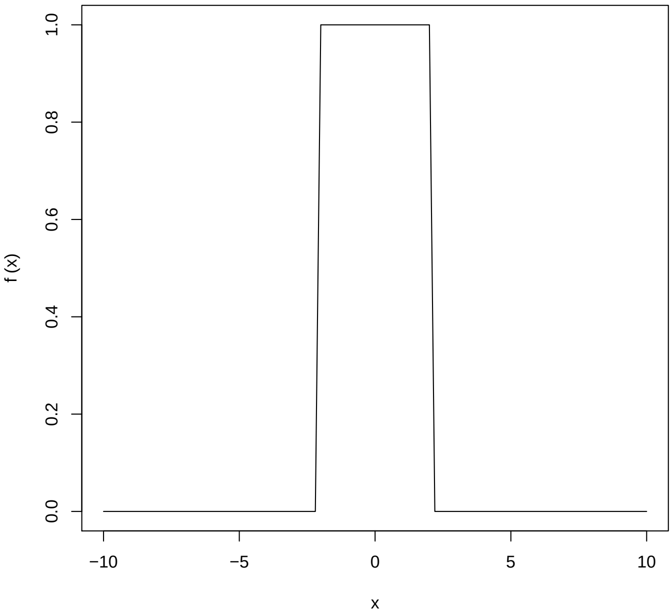


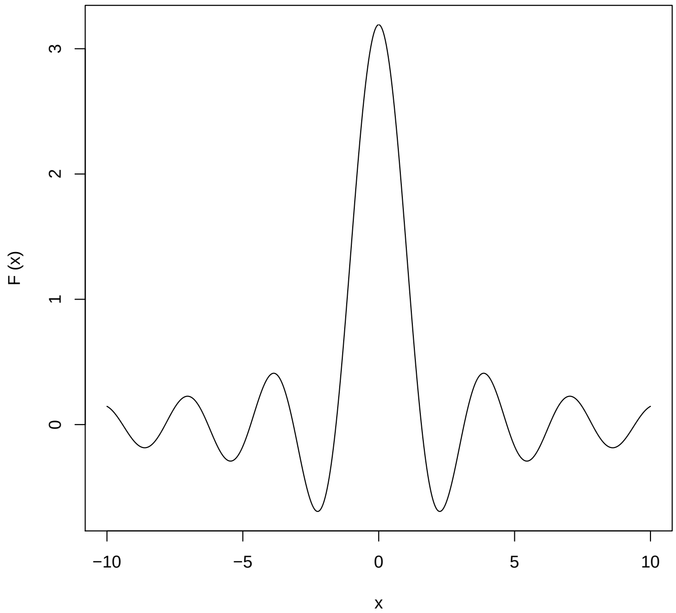


$$f(x) = \frac{1}{\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

$$F(x) = e^{-\frac{\sigma^2 x^2}{2}}$$

Fourier Transforms / Examples / Uniform

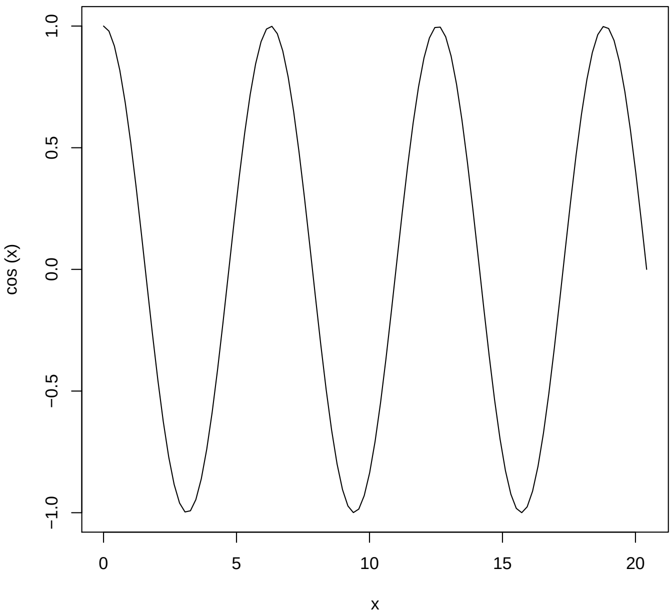


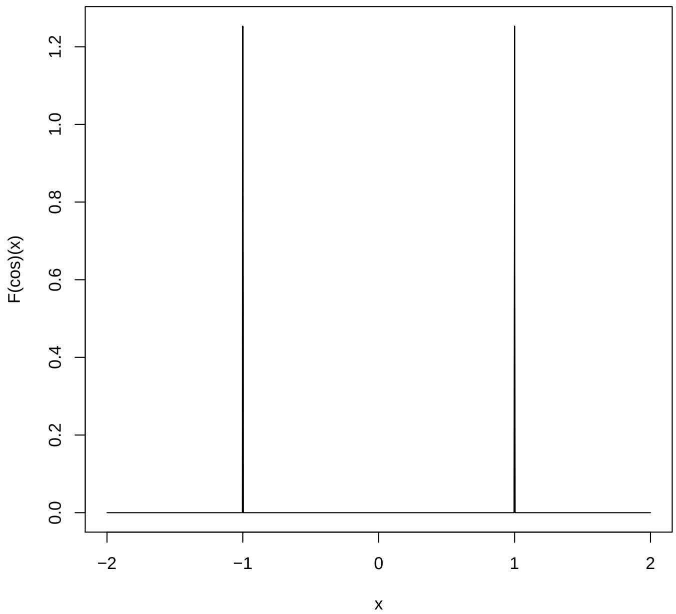


$$f(x) = \delta(x \in [-b, b])$$

$$F(x) = \frac{2b \sin(bx)}{\sqrt{2\pi}x}$$

Fourier Transforms / Examples / Cosine

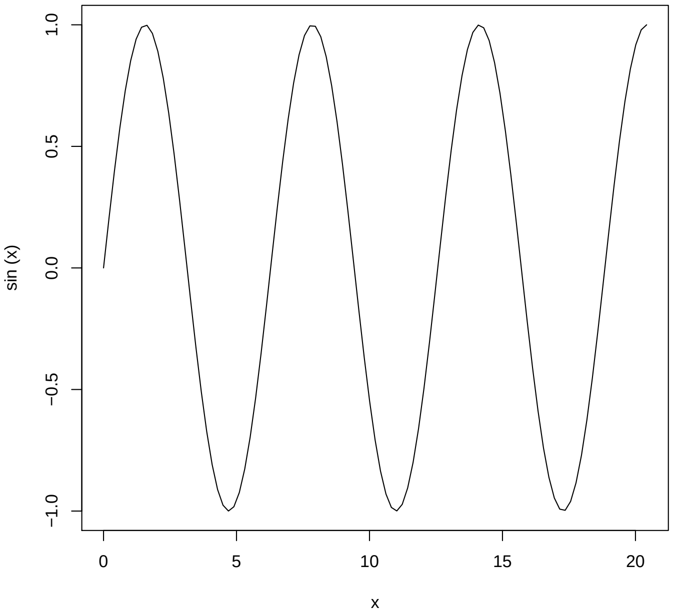


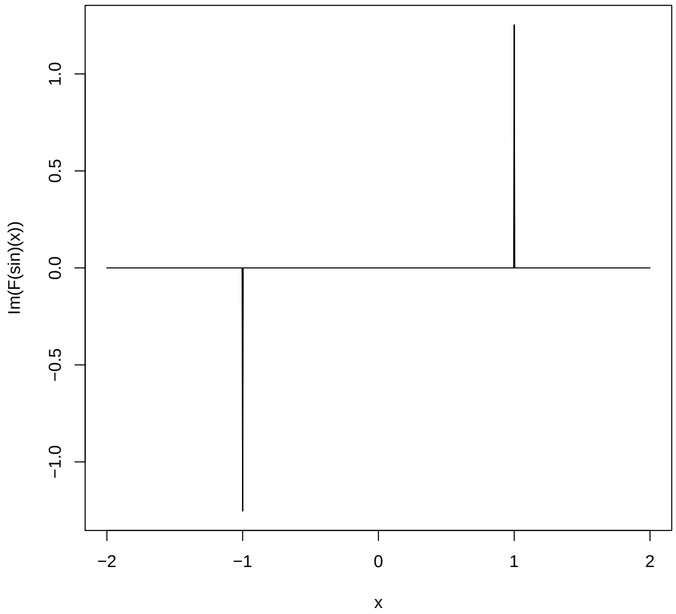


$$f(x) = \cos(\omega x)$$

$$F(x) = \sqrt{\frac{\pi}{2}}(\delta(x - \omega) + \delta(x + \omega))$$

Fourier Transforms / Examples / Sine





$$f(x) = \sin(\omega x)$$

$$F(x) = i \cdot \sqrt{\frac{\pi}{2}}(\delta(x-\omega) - \delta(x+\omega))$$

Properties

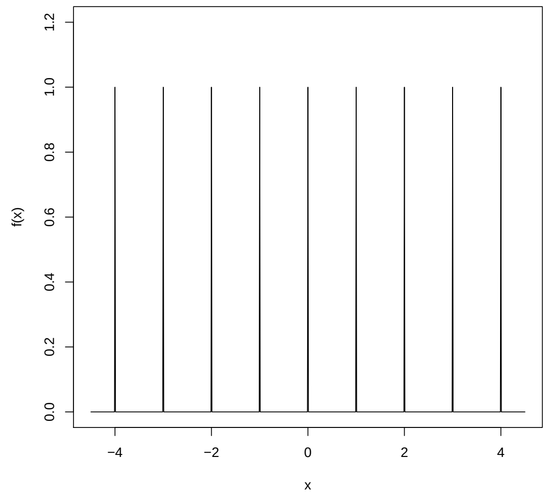
function h	Fourier transform H	property name
f	F	
F	$f(-x)$	inverse
$af + bg$	$aF + bG$	linearity
$f * g$	$F(x)G(x)$	convolution
$f(x)g(x)$	$\frac{1}{2\pi} F * G$	multiplication
$f(x-a)$	$e^{-iax}F(x)$	translation
$e^{iax}f(x)$	$F(x-a)$	modulation
$f(x/a)$	$\lvert a\rvert F(ax)$	scaling
f^*	$F^*(-x)$	complex conjugate
$f(x) \in \mathbb{R}$	$F(-x) = F^*(x)$	hermitian symmetry

Dirac Comb

The symbol

$$\Delta_T(x) := \sum_{n\in\mathbb{Z}} \delta(x - Tn)$$

is called **Dirac comb** (aka **impulse train**, **sampling function**, **Shah function**) with sampling interval T.



Lemma. The Fourier series of the Dirac comb Δ_T is

$$\Delta_T(x) = \frac{1}{T}\sum_{k\in\mathbb{Z}} e^{-i\frac{2\pi k}{T}x}$$

and its Fourier transform

$$\mathcal{F}(\Delta_T)(x) = \frac{1}{T}\cdot\Delta_{2\pi/T}(x) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\delta(x - \frac{2\pi}{T}n)$$

Dirac Comb

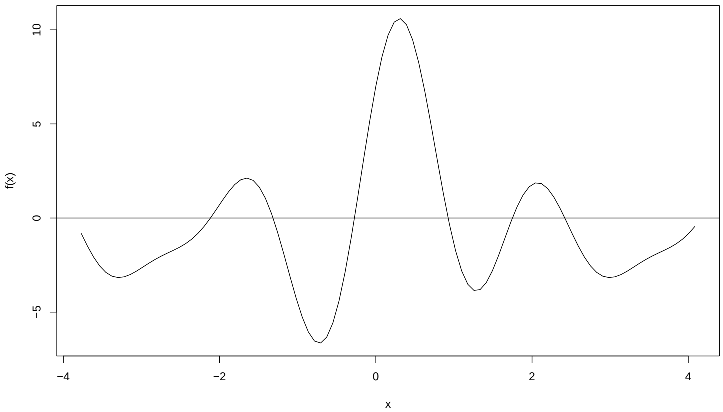
Proof: Obviously Δ_T is periodic with period T. Therefore

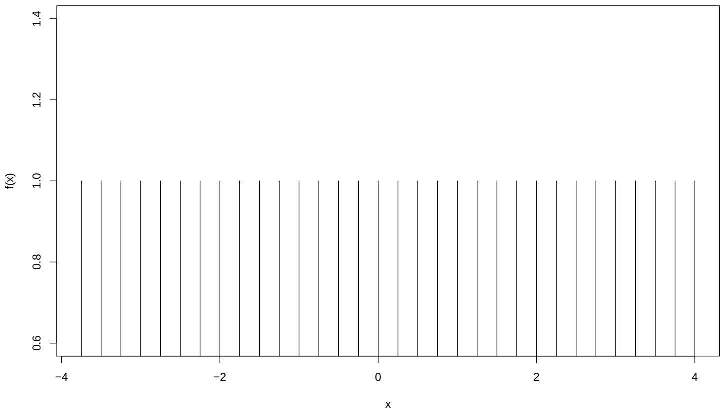
$$f(x) = \sum_{k \in \mathbb{Z}} c_k \, e^{-i\frac{2\pi k}{T}x}$$

with

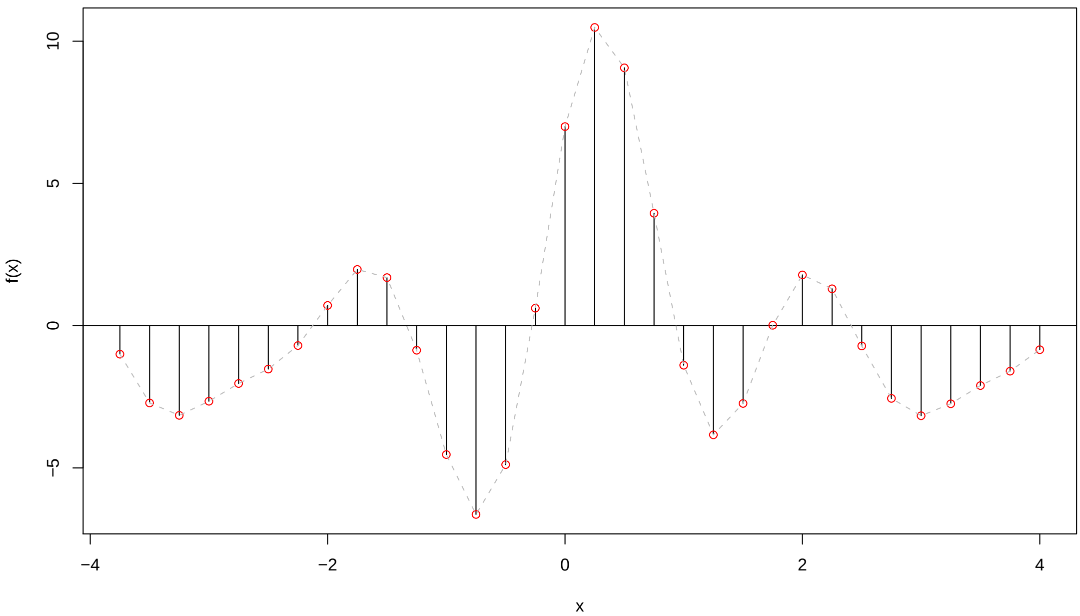
$$\begin{aligned}
c_k &= \frac{1}{T} \int_x^{x+T} \Delta_T(y) \, e^{-i\frac{2\pi k}{T}y} dy \\
&= \frac{1}{T} \int_{-T/2}^{+T/2} \Delta_T(y) \, e^{-i\frac{2\pi k}{T}y} dy \\
&= \frac{1}{T} \int_{-T/2}^{+T/2} \delta(y) \, e^{-i\frac{2\pi k}{T}y} dy \\
&= \frac{1}{T} e^{-i\frac{2\pi k}{T}0} \\
&= \frac{1}{T}
\end{aligned}$$

Sampling





Sampling



Sampling a function f at equidistant points $T \cdot \mathbb{Z}$ can be understood as

$$f^{\text{sampled}}(x) = f(x) \cdot \Delta_T(x)$$

Fourier Transform of a Sampled Function

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function and
f^{sampled} be a sample of f with sampling period T.

Then its Fourier transform $\mathcal{F}(f^{\text{sampled}})$ is $2\pi/T$-periodic and aggregates the Fourier transform $\mathcal{F}(f)$ over a $2\pi/T$-periodic grid:

$$\mathcal{F}(f^{\text{sampled}})(x) = \sum_{n \in \mathbb{Z}} \mathcal{F}(f)(x + n\frac{2\pi}{T})$$

If $\mathcal{F}(f)$ vanishes for $|x| > \pi/T$, then the Fourier transform $\mathcal{F}(f)$ is replicated in a period of the Fourier transform $\mathcal{F}(f^{\text{sampled}})$.

Otherwise replicas overlap and the Fourier transform becomes corrupted. This effect is called **aliasing**.

Fourier Transform of a Sampled Function

The maximal occurring frequency

$$\omega_{\max} := \max\{|x| \mid x \in \mathbb{R}, \mathcal{F}(f)(x) \neq 0\}$$

of the Fourier transform is called its **bandwidth**.

The frequency

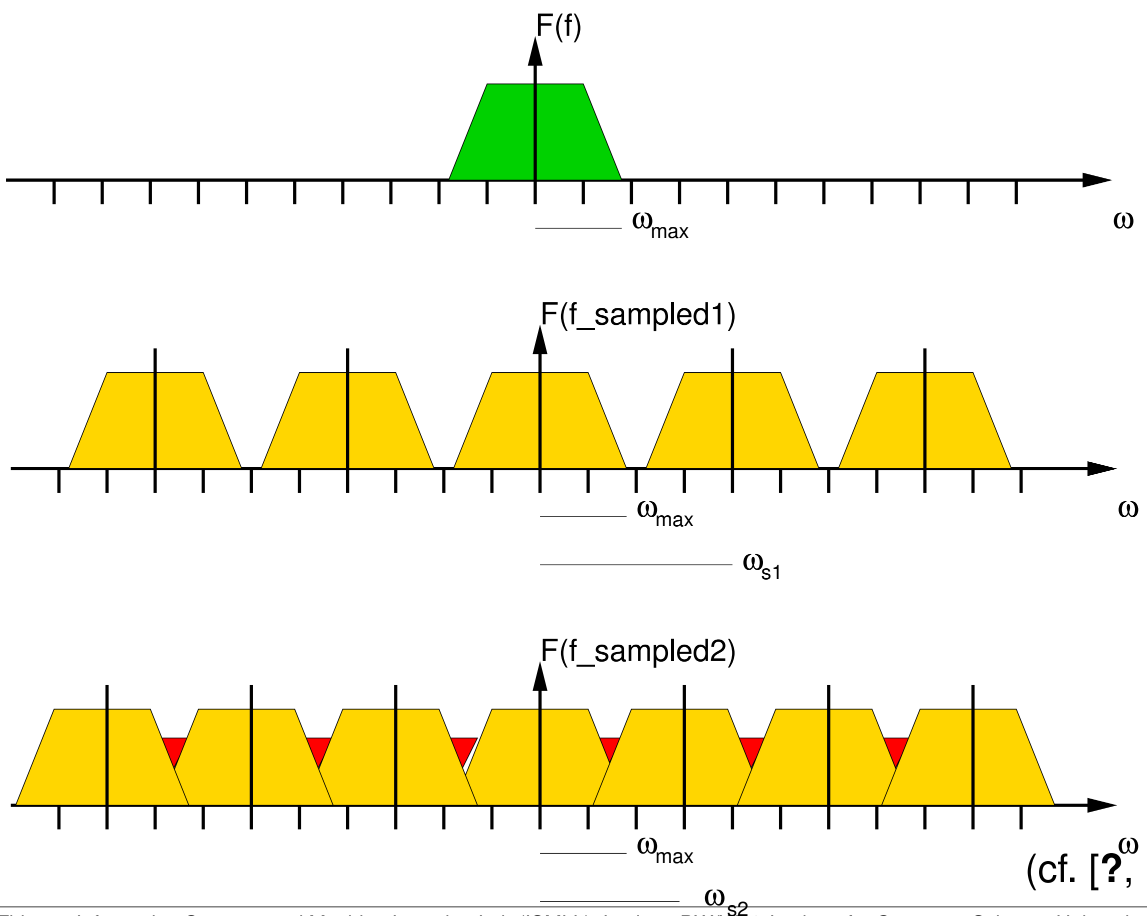
$$\omega_s := \frac{2\pi}{T}$$

of the sampling function is called its **sampling frequency**.

Then the sampling frequency must be at least twice the bandwith:

$$\omega_s > 2\omega_{\max}$$

Fourier Transform of a Sampled Function



(cf. [?, p. 330])

Fourier Transform of a Sampled Function

Proof.

$$\begin{aligned}
\mathcal{F}(f^{\text{sampled}})(x) =& \mathcal{F}(f \cdot \Delta_T)(x) \\
=& \mathcal{F}(f) * \mathcal{F}(\Delta_T)(x) \\
=& \mathcal{F}(f) * \frac{1}{T}\Delta_{2\pi/T}(x) \\
=& \mathcal{F}(f) * \frac{1}{T}\sum_{n\in\mathbb{Z}} \delta(x + n\frac{2\pi}{T}) \\
=& \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathcal{F}(f) * \delta(x + n\frac{2\pi}{T}) \\
=& \frac{1}{T}\sum_{n\in\mathbb{Z}} \mathcal{F}(f)(x + n\frac{2\pi}{T})
\end{aligned}$$

If $|x| < \pi/T$ and $\mathcal{F}(f)(y)$ vanishes for $|y| > \pi/T$, then

$$\mathcal{F}(f^{\text{sampled}})(x) = \frac{1}{T}\mathcal{F}(f)(x)$$

The Fourier Transform of a Discrete Function

Let f be a discrete function:

$$f(x) = \sum_{n\in\mathbb{Z}} y_n\, \delta(x-n), \quad y_n \in \mathbb{R}$$

Then its Fourier transform is:

$$\begin{aligned}
\mathcal{F}(f)(\omega) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\
=& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} y_n\, \delta(x-n)\, e^{-i\omega x} dx \\
=& \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} y_n\, e^{-i\omega x}\, \delta(x-n) dx \\
=& \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} y_n\, e^{-i\omega n} \\
=& \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} f(n)\, e^{-i\omega n}
\end{aligned}$$

i.e., a periodic function — or equivalently: a function defined on an interval.

Definition

Let $f : \{0, 1, \ldots, N-1\} \to \mathbb{C}$ be a finite discrete function, then

$$\begin{aligned} F : \{0, 1, \ldots, N-1\} &\to \mathbb{C} \\ \omega &\mapsto \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)(\cos(2\pi \frac{\omega x}{N}) - i \sin(2\pi \frac{\omega x}{N})) \\ &= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) e^{-i2\pi \frac{\omega x}{N}} \end{aligned}$$

is called **discrete Fourier transform of f**, denoted DFT(f).

Then

$$\begin{aligned} f(x) =& \frac{1}{\sqrt{N}} \sum_{\omega=0}^{N-1} F(\omega)(\cos(2\pi \frac{\omega x}{N}) + i \sin(2\pi \frac{\omega x}{N})) \\ =& \frac{1}{\sqrt{N}} \sum_{\omega=0}^{N-1} F(\omega) e^{i2\pi \frac{\omega x}{N}} \end{aligned}$$

This is called **inverse discrete Fourier transform of f**, denoted DFT$^{-1}(F)$..

Example

x	$f(x)$		ω	$F(\omega)$
0	$1+0i$		0	$14.2302+0.0000i$
1	$3+0i$	DFT	1	$-5.6745-2.9198i$
2	$5+0i$	$\longrightarrow$	2	$0.0000+0.0000i$
3	$7+0i$		3	$-0.0176-0.6893i$
4	$9+0i$		4	$0.0000+0.0000i$
5	$8+0i$		5	$0.3162+0.0000i$
6	$6+0i$		6	$0.0000+0.0000i$
7	$4+0i$	DFT^{-1}	7	$-0.0176+0.6893i$
8	$2+0i$	$\longleftarrow$	8	$0.0000+0.0000i$
9	$0+0i$		9	$-5.6745+2.9198i$

(cf. [?, p. 333])

$$\text{DFT}(f)(0) = \frac{1}{\sqrt{10}} \sum_{x=0}^{9} f(x)(\cos(2\pi\frac{0x}{10}) - i\sin(2\pi\frac{0x}{10})) = \frac{1}{\sqrt{10}} \sum_{x=0}^{9} f(x) = \frac{45}{\sqrt{10}} = 14.2303$$

Example

x	$f(x)$		ω	$F(\omega)$
0	$1+0i$		0	$14.2302+0.0000i$
1	$3+0i$	DFT	1	$-5.6745-2.9198i$
2	$5+0i$	$\longrightarrow$	2	$0.0000+0.0000i$
3	$7+0i$		3	$-0.0176-0.6893i$
4	$9+0i$		4	$0.0000+0.0000i$
5	$8+0i$		5	$0.3162+0.0000i$
6	$6+0i$		6	$0.0000+0.0000i$
7	$4+0i$	DFT^{-1}	7	$-0.0176+0.6893i$
8	$2+0i$	$\longleftarrow$	8	$0.0000+0.0000i$
9	$0+0i$		9	$-5.6745+2.9198i$

(cf. [?, p. 333])

$$\begin{aligned}\text{DFT}(f)(1) =& \frac{1}{\sqrt{10}} \sum_{x=0}^{9} f(x)(\cos(2\pi\frac{1x}{10}) - i\sin(2\pi\frac{1x}{10}))\\ =& \frac{1}{\sqrt{10}} \sum_{x=0}^{9} f(x)\cos(\pi\frac{x}{5}) - i\frac{1}{\sqrt{10}} \sum_{x=0}^{9} f(x)\sin(\pi\frac{x}{5}) = -5.6745 - 2.9198i\end{aligned}$$

Discrete Fourier Transform / Algorithm (naive)

For $x \in \mathbb{C}$, denote $\Re(x)$ its **real part** and $\Im(x)$ its **imaginary part**, i.e.,

$$x = \Re(x) + i\Im(x), \quad \Re(x), \Im(x) \in \mathbb{R}$$

(for $\Re$ one often also uses Re, for $\Im$ also Im).

To compute the discrete Fourier transform, one computes dft$(f)(\omega)$ for $\omega = 0, \ldots, N-1$ via:

$$\begin{aligned}
\text{DFT}(f)(\omega) =& \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)(\cos(2\pi \frac{\omega x}{N}) - i \sin(2\pi \frac{\omega x}{N})) \\
=& \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (\Re(f(x)) + i\Im(f(x)))(\cos(2\pi \frac{\omega x}{N}) - i \sin(2\pi \frac{\omega x}{N})) \\
=& \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \Re(f(x)) \cos(2\pi \frac{\omega x}{N}) + \Im(f(x)) \sin(2\pi \frac{\omega x}{N}) \\
&+ i \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} -\Re(f(x)) \sin(2\pi \frac{\omega x}{N}) + \Im(f(x)) \cos(2\pi \frac{\omega x}{N})
\end{aligned}$$

Discrete Fourier Transform / Algorithm (naive)

```
dft-naive(sequence f = (f(x)_0, f(x)_1)_{x=0,...,N-1}) :
N := length(f)
F := (F(x)_0, F(x)_1)_{x=0,...,N-1} = (0, 0)_{x=0,...,N-1}
for ω := 0, ..., N − 1 do
    c := (c_0, c_1) := (0, 0)
    for x := 0, ..., N − 1 do
        c_0 := c_0 + f(x)_0 · cos(2πωx/N) + f(x)_1 · sin(2πωx/N)
        c_1 := c_1 − f(x)_0 · sin(2πωx/N) + f(x)_1 · cos(2πωx/N)
    od
    c_0 := c_0/√N
    c_1 := c_1/√N
    F(ω) := c
od
return F
```

Discrete Fourier Transform / Algorithm (naive)

When computing the values of the discrete Fourier transform for different arguments ω, the cosine and sine functions repeatedly are callled with the same arguments:

dft$(f)(1)$:

$$\begin{aligned}
&\cos(2\pi \cdot 0/10)\\
&\cos(2\pi \cdot 1/10)\\
&\cos(2\pi \cdot 2/10)\\
&\cos(2\pi \cdot 3/10)\\
&\cos(2\pi \cdot 4/10)\\
&\cos(2\pi \cdot 5/10)\\
&\cos(2\pi \cdot 6/10)\\
&\cos(2\pi \cdot 7/10)\\
&\cos(2\pi \cdot 8/10)\\
&\cos(2\pi \cdot 9/10)
\end{aligned}$$

dft$(f)(2)$:

$$\begin{aligned}
&\cos(2\pi \cdot 0/10)\\
&\cos(2\pi \cdot 2/10)\\
&\cos(2\pi \cdot 4/10)\\
&\cos(2\pi \cdot 6/10)\\
&\cos(2\pi \cdot 8/10)\\
&\cos(2\pi \cdot 10/10) = \cos(2\pi \cdot 0/10)\\
&\cos(2\pi \cdot 12/10) = \cos(2\pi \cdot 2/10)\\
&\cos(2\pi \cdot 14/10) = \cos(2\pi \cdot 4/10)\\
&\cos(2\pi \cdot 16/10) = \cos(2\pi \cdot 6/10)\\
&\cos(2\pi \cdot 18/10) = \cos(2\pi \cdot 8/10)
\end{aligned}$$

Caching the expensive sine and cosine computations accelerates the algorithm!

Discrete Fourier Transform / Algorithm (naive, cached)

dft-naive-cached(sequence $f = (f(x)_0, f(x)_1)_{x=0,\ldots,N-1})$:
$N :=$ length(f)
for $\omega := 0, \ldots, N-1$ **do**
    $C(\omega) := \cos(2\pi\omega/N)$
    $S(\omega) := \sin(2\pi\omega/N)$
od
$F := (F(x)_0, F(x)_1)_{x=0,\ldots,N-1} = (0,0)_{x=0,\ldots,N-1}$
for $\omega := 0, \ldots, N-1$ **do**
    $c := (c_0, c_1) := (0,0)$
    for $x := 0, \ldots, N-1$ **do**
        $c_0 := c_0 + f(x)_0 \cdot C(\omega x \bmod N) + f(x)_1 \cdot S(\omega x \bmod N)$
        $c_1 := c_1 - f(x)_0 \cdot S(\omega x \bmod N) + f(x)_1 \cdot C(\omega x \bmod N)$
    od
    $c_0 := c_0/\sqrt{N}$
    $c_1 := c_1/\sqrt{N}$
    $F(\omega) := c$
od
return F

Fast Fourier Transform (Gauss ca. 1805; Cooley/Tukey 1965)

The naive algorithm for DFT still has complexity $O(N^2)$.

The **Fast Fourier Transform** algorithm is based on a decomposition of the DFT for sequences f of even length N:

$$\text{DFT}(f)(\omega) = \text{DFT}(f^{\text{even}})(\omega \bmod N/2) + e^{-i2\pi\omega/N}\text{DFT}(f^{\text{odd}})(\omega \bmod N/2)$$

for $\omega = 0, \ldots, N-1$
and where

$$f^{\text{even}}(x) := f(2x), \quad x = 0, \ldots, N/2$$
$$f^{\text{odd}}(x) := f(2x+1)$$

Fast Fourier Transform / Proof

Proof.

$$\begin{aligned}
\text{DFT}(f)(\omega) &= \sum_{x=0}^{N-1} f(x) e^{-i2\pi\frac{\omega x}{N}} \\
&= \sum_{x=0 \text{ even}}^{N-1} f(x) e^{-i2\pi\frac{\omega x}{N}} + \sum_{x=0 \text{ odd}}^{N-1} f(x) e^{-i2\pi\frac{\omega x}{N}} \\
&= \sum_{x=0}^{N/2-1} f(2x) e^{-i2\pi\frac{\omega 2x}{N}} + \sum_{x=0}^{N/2-1} f(2x+1) e^{-i2\pi\frac{\omega(2x+1)}{N}} \\
&= \sum_{x=0}^{N/2-1} f^{\text{even}}(x) e^{-i2\pi\frac{\omega x}{N/2}} + e^{-i2\pi\omega/N} \sum_{x=0}^{N/2-1} f^{\text{odd}}(x) e^{-i2\pi\frac{\omega x}{N/2}} \\
&= \text{DFT}(f^{\text{even}})(\omega) + e^{-i2\pi\omega/N}\text{DFT}(f^{\text{odd}})(\omega)
\end{aligned}$$

or more exaclty, as $\text{DFT}(f^{\text{even}})(\omega)$ is only defined for $\omega < N/2$:

$$= \text{DFT}(f^{\text{even}})(\omega \bmod N/2) + e^{-i2\pi\omega/N}\text{DFT}(f^{\text{odd}})(\omega \bmod N/2)$$

Fast Fourier Transform / Real version

$$\begin{aligned}\text{DFT}(f)(\omega) =& \text{DFT}(f^{\text{even}})(\omega \bmod N/2) + e^{-i2\pi\omega/N}\text{DFT}(f^{\text{odd}})(\omega \bmod N/2)\\ =& \text{DFT}(f^{\text{even}})(\omega \bmod N/2)\\ &+ (\cos 2\pi\omega/N - i \sin 2\pi\omega/N)\text{DFT}(f^{\text{odd}})(\omega \bmod N/2)\end{aligned}$$

and thus

$$\begin{aligned}\Re(\text{DFT}(f)(\omega)) =& \Re(\text{DFT}(f^{\text{even}})(\omega \bmod N/2))\\ &+ \cos 2\pi\omega/N \cdot \Re(\text{DFT}(f^{\text{odd}})(\omega \bmod N/2))\\ &+ \sin 2\pi\omega/N \cdot \Im(\text{DFT}(f^{\text{odd}})(\omega \bmod N/2))\end{aligned}$$

$$\begin{aligned}\Im(\text{DFT}(f)(\omega)) =& \Im(\text{DFT}(f^{\text{even}})(\omega \bmod N/2))\\ &+ \cos 2\pi\omega/N \cdot \Im(\text{DFT}(f^{\text{odd}})(\omega \bmod N/2))\\ &- \sin 2\pi\omega/N \cdot \Re(\text{DFT}(f^{\text{odd}})(\omega \bmod N/2))\end{aligned}$$

Image Analysis / 4. Discrete Fourier Transform

Fast Fourier Transform / Algorithm

fft(sequence $f = (f(x)_0, f(x)_1)_{x=0,\ldots,N-1})$:
$N := \text{length}(f)$
if N is even
  $F := (F(x)_0, F(x)_1)_{x=0,\ldots,N-1} = (0,0)_{x=0,\ldots,N-1}$
  $A := \text{fft}((f(x))_{x=0,2,4,\ldots,N-2})$
  $B := \text{fft}((f(x))_{x=1,3,5,,\ldots,N-1})$
  for $\omega := 0, \ldots, N-1$ **do**
    $a := A(\omega \bmod N/2)$
    $b := B(\omega \bmod N/2)$
    $F(\omega)_0 := a_0 + \cos 2\pi\omega/N \cdot b_0 + \sin 2\pi\omega/N \cdot b_1$
    $F(\omega)_1 := a_1 + \cos 2\pi\omega/N \cdot b_1 - \sin 2\pi\omega/N \cdot b_0$
  od
  return
else
  $F := \text{dft-naive-cached}(f)$
fi
return F

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Course on Image Analysis, winter term 2011/12

Fast Fourier Transform / Outlook

- The computation of $\cos 2\pi\omega/N$ and $\sin 2\pi\omega/N$ also can be done recursively using the addition formulas.
- In this way, FFT best is applied to sequences of length 2^n (called **radix-2 case**).
- The FFT decomposition works with any factorization $N = N_1 \cdot N_2$ in a similar way, and thus also for sequences of length other than 2^n.
- FFT has complexity $O(N \log N)$ (if N is a power of 2).
- An early experiment from 1969 reports a runtime of 13 1/2 hours for computing the DFT of a sequence of length 2048 by the naive method and 2.4 seconds using FFT.
- In practice, FFT is implemented in a linearized version avoiding explicit recursions (see [**?**, p. 839]).

Types of Fourier Transforms

type of function f	sup f	sup $\mathcal{F}(f)$	type of Fourier decomp.
general (integrable) function	$\mathbb{R}$	$\mathbb{R}$	(general) Fourier decomposition
periodic function, function on interval	interval $I \subseteq \mathbb{R}$	$\mathbb{Z}$	Fourier series
general (integrable) discrete function (sum of Diracs)	$\mathbb{Z}$	interval $I \subseteq \mathbb{R}$	discrete-time Fourier transform
periodic discrete function, finite discrete function (finite sum of Diracs)	finite $I \subseteq \mathbb{Z}$	I	discrete Fourier transform

General Fourier Transform in 2D

For two-dimensional functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ Fourier Transforms, Fourier Series and Discrete Fourier Transforms can be defined analogously.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function (that satisfies some regularity conditions). Then

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R} &\to \mathbb{C} \\ (\omega_1, \omega_2) &\mapsto \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\, e^{-i\omega_1 x}\, e^{-i\omega_2 y}\, dy\, dx \end{aligned}$$

exists for each (ω_1, ω_2) and is a continuous function called **Fourier Transform of f** (aka **Fourier spectrum of f**).

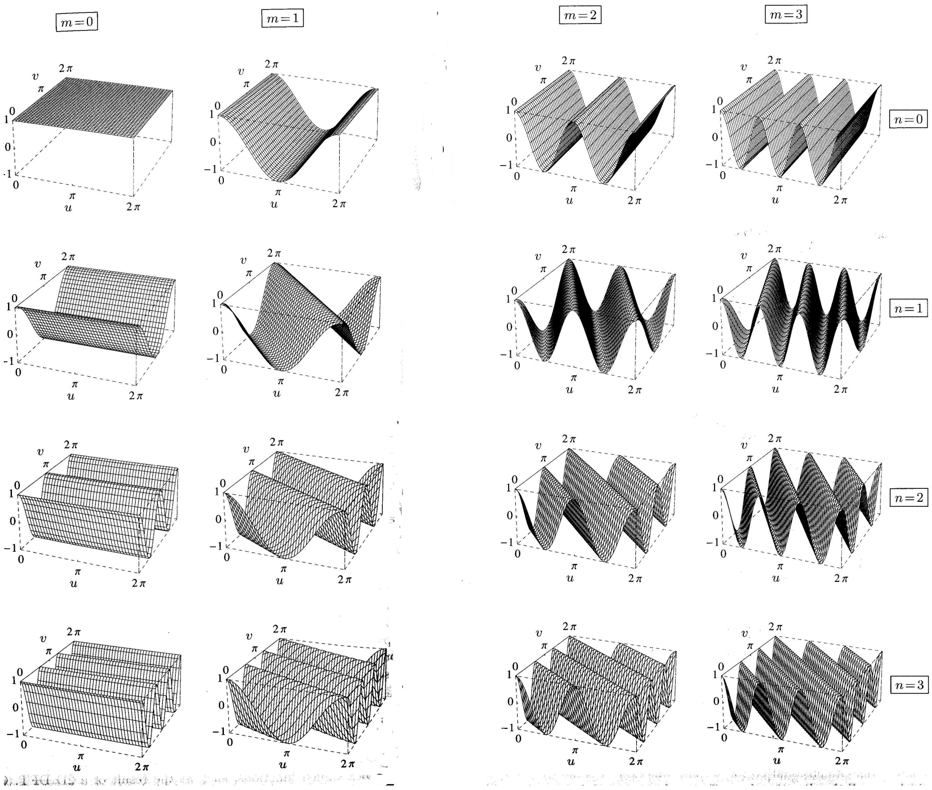
One can show that (if F also satisfies some regularity conditions):

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega_1, \omega_2) e^{i\omega_1 x} e^{i\omega_2 y} d\omega_1 d\omega_2$$

This is called **Inverse Fourier Transform**.

We will write $\mathcal{F}(f) := F$ for the Fourier transform of a function f and $\mathcal{F}^{-1}(F)$ for the inverse Fourier transform of a function F.

Bases in 2D



Fourier Series in 2D

If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is (T_1, T_2)-periodic, i.e.,

$$f(x,y) = f(x+T_1, y+T_2) \quad \forall x, y \in \mathbb{R}$$

then f can already be reconstructed from $\mathbb{Z}$-many Fourier coefficients:

$$f(x,y) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} F(n,m)\, e^{i\frac{2\pi}{T_1}nx}\, e^{i\frac{2\pi}{T_2}my}$$

with

$$F(n,m) := \frac{1}{2\pi} \int_{-T_1/2}^{+T_1/2} \int_{-T_2/2}^{+T_2/2} f(x,y)\, e^{-in\omega_1 x}\, e^{-im\omega_2 y}\, dy\, dx$$

Discrete Fourier Transform in 2D

If f is discrete, i.e.,

$$f(x,y) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} y_{n,m}\delta(x - T_1 n, y - T_2 m), \quad y_{n,m} \in \mathbb{R}$$

then its Fourier transform is periodic:

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega_1,\omega_2)\, e^{i\omega_1 x}\, e^{i\omega_2 y}\, d\omega_1\, d\omega_2$$

with

$$F(\omega_1,\omega_2) := \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} f(n,m)\, e^{-i\omega_1 n}\, e^{-i\omega_2 m}$$

Discrete Fourier Transform in 2D

And finally, if f is discrete and finite, i.e.,

$$f(x,y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} y_{n,m}\, \delta(x - n, y - m), \quad y_{n,m} \in \mathbb{R}$$

then its Fourier transform is periodic and made from finitely many components:

$$f(x,y) = \frac{1}{\sqrt{NM}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} F(\omega_1,\omega_2)\, e^{i\omega_1 x}\, e^{i\omega_2 y}$$

with

$$F(\omega_1,\omega_2) := \frac{1}{\sqrt{NM}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(n,m)\, e^{-i\omega_1 n}\, e^{-i\omega_2 m}$$

Discrete Fourier Transform in 2D / Algorithm (naive, cached)

dft-2d-naive-cached(array $f = (f(x)_0, f(x)_1)_{x=0,\ldots,N-1,y=0,\ldots,M-1})$:
for $\omega := 0, \ldots, N-1$ **do**
  $C_1(\omega) := \cos(2\pi\omega/N)$
  $S_1(\omega) := \sin(2\pi\omega/N)$
od
for $\omega := 0, \ldots, M-1$ **do**
  $C_2(\omega) := \cos(2\pi\omega/M)$
  $S_2(\omega) := \sin(2\pi\omega/M)$
od
$F := (F(x)_0, F(x)_1)_{x=0,\ldots,N-1,y=0,\ldots,M-1} = (0,0)_{x=0,\ldots,N-1,y=0,\ldots,M-1}$
for $\omega_1 := 0, \ldots, N-1$ **do**
  for $\omega_2 := 0, \ldots, M-1$ **do**
    $c := (c_0, c_1) := (0,0)$
    for $x := 0, \ldots, N-1$ **do**
      for $y := 0, \ldots, M-1$ **do**
        $C := C_1(\omega_1 x \bmod N) \cdot C_2(\omega_2 y \bmod M) - S_1(\omega_1 x \bmod N) \cdot S_2(\omega_2 y \bmod M)$
        $S := S_1(\omega_1 x \bmod N) \cdot C_2(\omega_2 y \bmod M) + C_1(\omega_1 x \bmod N) \cdot S_2(\omega_2 y \bmod M)$
        $c_0 := c_0 + f(x)_0 \cdot C - f(x)_1 \cdot S$
        $c_1 := c_1 - f(x)_0 \cdot S + f(x)_1 \cdot C$
      od
    od
    $c_0 := c_0/\sqrt{NM}$
    $c_1 := c_1/\sqrt{NM}$
    $F(\omega) := c$
  od
od
return F

Discrete Fourier Transform in 2D / Separability

$$
\begin{aligned}
F(\omega_1, \omega_2) :=& \frac{1}{\sqrt{NM}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(n,m)\, e^{-i\omega_1 n}\, e^{-i\omega_2 m} \\
=& \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(\frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f(n,m)\, e^{-i\omega_2 m} \right) e^{-i\omega_1 n} \\
=& \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \text{DFT}((f(n,m))_{m=1,\ldots,M})(\omega_2)\, e^{-i\omega_1 n}
\end{aligned}
$$

Discrete Fourier Transform in 2D / FFT

fft-2d(array $f = (f(x)_0, f(x)_1)_{x=0,\ldots,N-1,y=0,\ldots,M-1})$:
$G := (G(x)_0, G(x)_1)_{x=0,\ldots,N-1,y=0,\ldots,M-1} = (0,0)_{x=0,\ldots,N-1,y=0,\ldots,M-1}$
for $\omega_1 := 0, \ldots, N-1$ **do**
  $G(\omega_1, .) :=$ fft$(f(\omega_1, y)_{y=0,\ldots,M-1})$
od
$F := (F(x)_0, F(x)_1)_{x=0,\ldots,N-1,y=0,\ldots,M-1} = (0,0)_{x=0,\ldots,N-1,y=0,\ldots,M-1}$
for $\omega_2 := 0, \ldots, M-1$ **do**
  $F(., \omega_2) :=$ fft$(G(x, \omega_2)_{x=0,\ldots,N-1})$
od
return F

Power Spectrum

The fourier spectrum of

– a discrete $N \times M$ gray-scale image f, i.e., with one channel,

is

– a discrete $N \times M$ image F with complex intensity values, i.e., two channels.

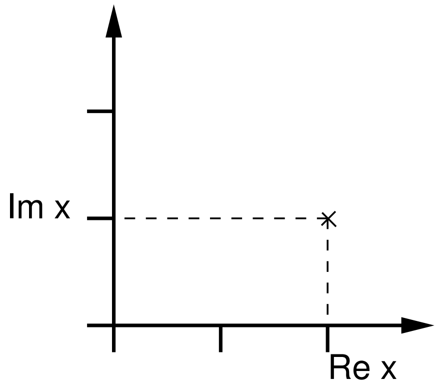
For visualization one usually shows the **power spectrum** defined as:

$$\mathcal{F}^{\text{power}}(f)(x) := \sqrt{(\Re\mathcal{F}(f)(x))^2 + (\Im\mathcal{F}(f)(x))^2}$$

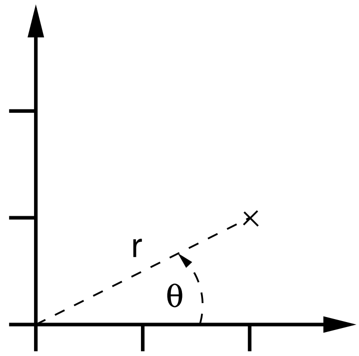
The power spectrum measures the absolute value of the complex amplitude.
The complementary information θ called **phase** is not shown.

What does Power Spectrum mean? — Complex Coordinates

Cartesian coordinates:



Polar coordinates:



$$x = (\Re x, \Im x) \in \mathbb{R} \times \mathbb{R}$$

$$\Re x = r\cos\theta$$
$$\Im x = r\sin\theta$$

$$x = (r, \theta) \in \mathbb{R}_0^+ \times [0, 2\pi)$$

$$r = \sqrt{(\Re x)^2 + (\Im x)^2}$$
$$\theta = \begin{cases} \arctan(\Re x / \Im x), & \text{if } x > 0, y > 0 \\ \arctan(\Re x / \Im x) + \pi, & \text{if } y < 0 \\ \arctan(\Re x / \Im x) + 2\pi, & \text{if } x > 0, y < 0 \end{cases}$$

$$x = \Re x + i \Im x = r\cos\theta + ir\sin\theta = re^{i\theta}$$

Example

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:

Displaying Fourier Power Spectra

For displaying spectra, some further conventions are used:

- As the scale of many power spectra is dominated by a few large values, one usually plots

$$\log \mathcal{F}^{\text{power}}(f)(x) \quad \text{or} \quad (\mathcal{F}^{\text{power}}(f)(x))^{\frac{1}{2}}$$

  instead of the raw power spectrum values $\mathcal{F}^{\text{power}}(f)(x)$.

- Usually, the **centered spectrum** is shown, i.e., the intensities for

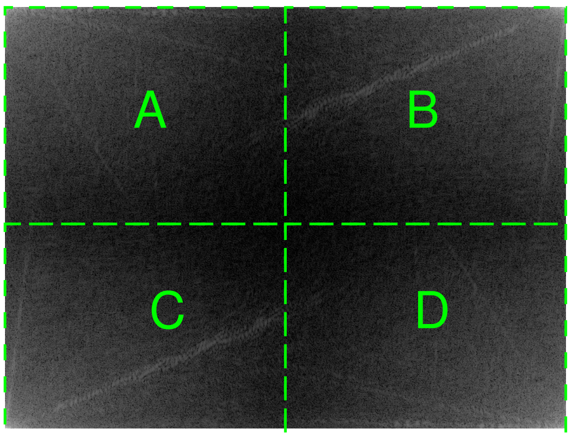
$$x \in \{-N/2, -N/2+1, \ldots, -1, 0, 1, \ldots, N/2-1, N/2\}$$

  and

$$y \in \{-M/2, -M/2+1, \ldots, -1, 0, 1, \ldots, M/2-1, M/2\}$$

  instead of the intensities for $0, 1, \ldots, N-1$ and $0, 1, \ldots, M-1$.

Centered Spectrum



Centered Spectrum

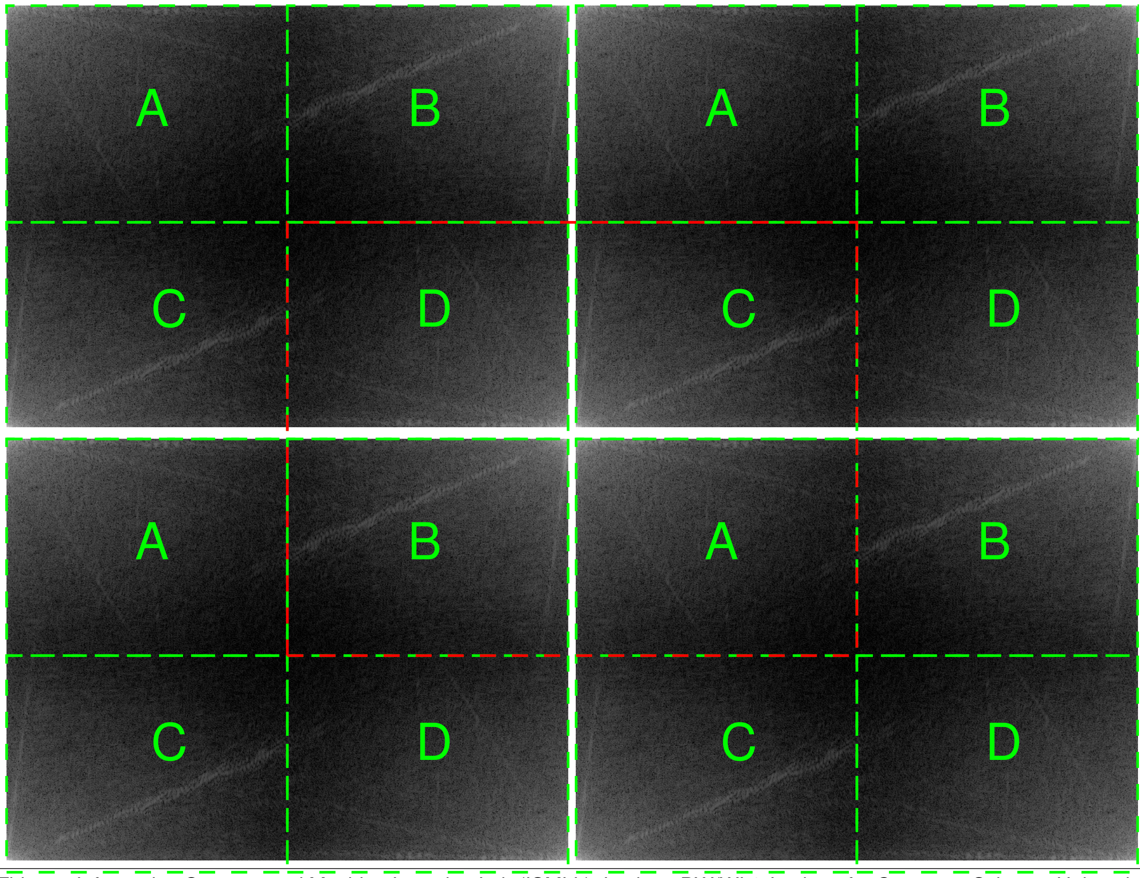
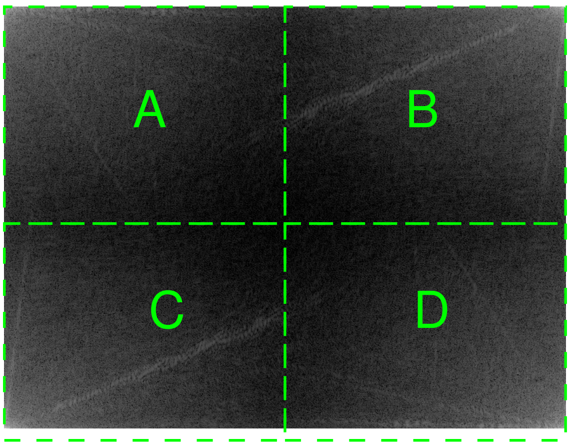


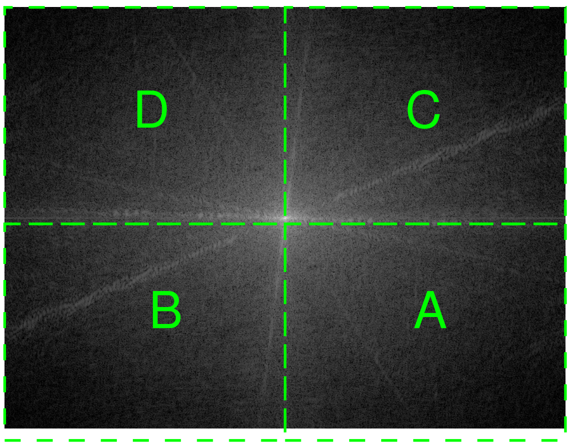
Image Analysis / 5. Two-dimensional Fourier Transforms

Centered Spectrum

original spectrum:



centered spectrum:



Lars Schmidt-Thieme, Information Systems and Machine Learning Lab (ISMLL), Institute BW/WI & Institute for Computer Science, University of Hildesheim
Course on Image Analysis, winter term 2011/12
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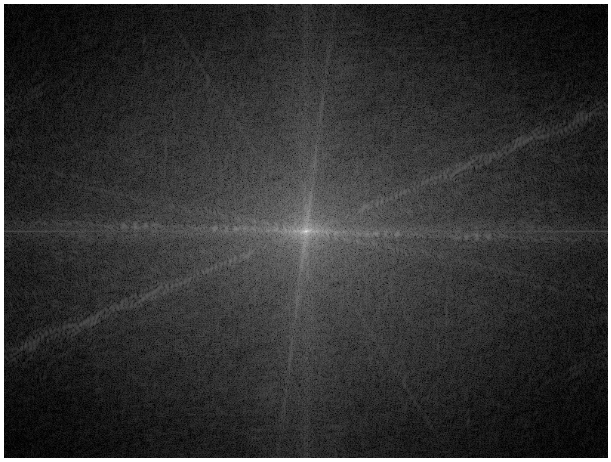
Example

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



Symmetry of Fouier Power Spectra for Real Images

Usually images are real, i.e., $f(x) \in \mathbb{R}$ (not $\mathbb{C}$).

For real functions, we know that

$$\mathcal{F}(f)(-x) = \mathcal{F}^*(f)(x)$$

is hermitian (with $x^* := \Re x - i\,\Im x$).

As $x \in \mathbb{C}$ has the same radius as x^*:

$$r(x) = \sqrt{(\Re x)^2 + (\Im x)^2} \stackrel{?}{=} r(x^*) = r(\Re x - i\Im x) = \sqrt{(\Re x)^2 + (-\Im x)^2}$$

for real functions $\mathcal{F}(f)$ and $\mathcal{F}^*(f)$ have the same radius and thus

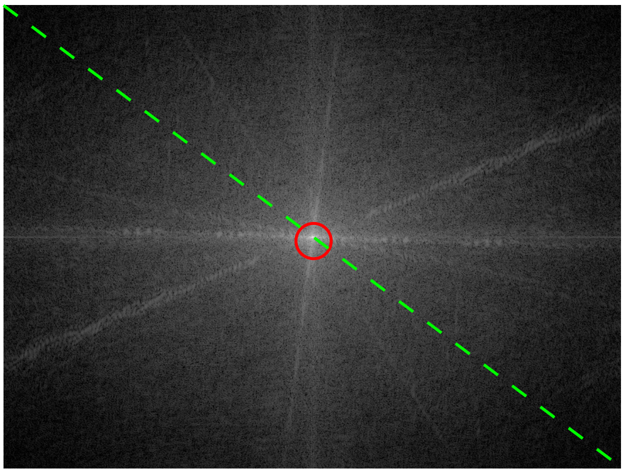
$$\mathcal{F}^{\text{power}}(f)(-x) = \mathcal{F}^{\text{power}}(f)(x)$$

i.e., the power spectrum is symmetric around the origin.

Symmetry of Fouier Power Spectra for Real Images

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



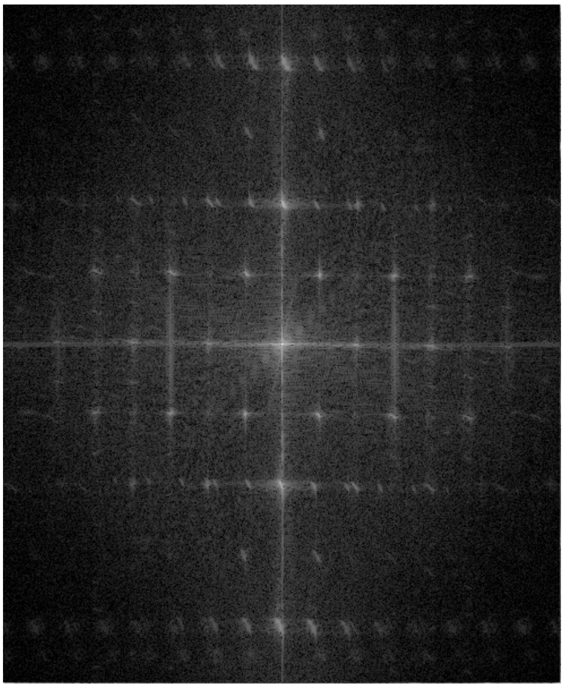
Spectra of real images are symmetric around the origin (red circle).

So storing just half of the power spectrum is sufficient
(e.g., above green line — any line through the origin will do).

Another Example

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



Low Pass Filters

High frequencies are responsible for sharp edges,
low frequencies for constant and slowly changing areas.

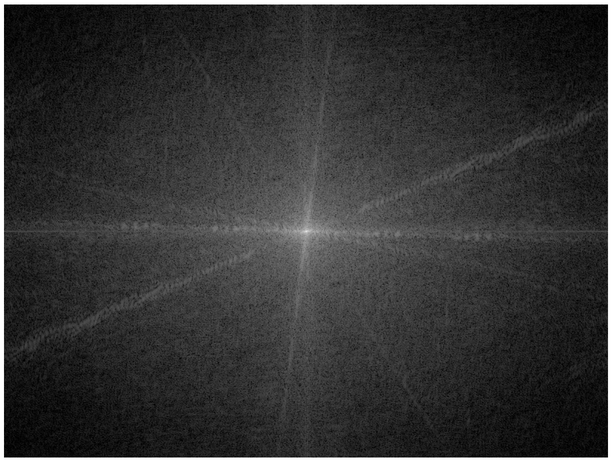
Low pass filters

– retain only low frequencies $\omega \leq \omega_{\text{max}}$,
– i.e., filter out high frequencies.
– and thus smooth / blur an image.

Low Pass Filters / Example

image f:

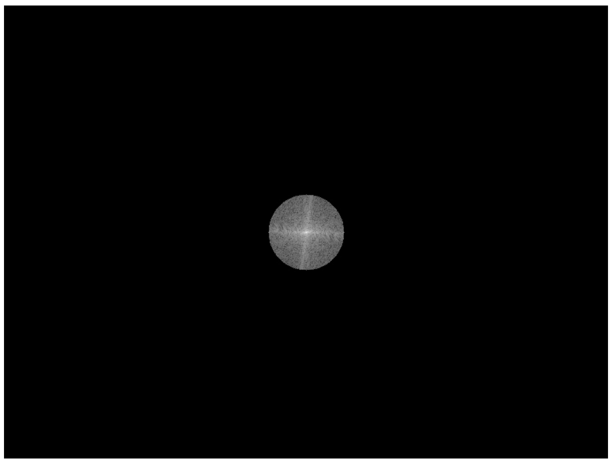
power spectrum $\mathcal{F}^{\text{power}}(f)$:



Low Pass Filters / Example

image f:

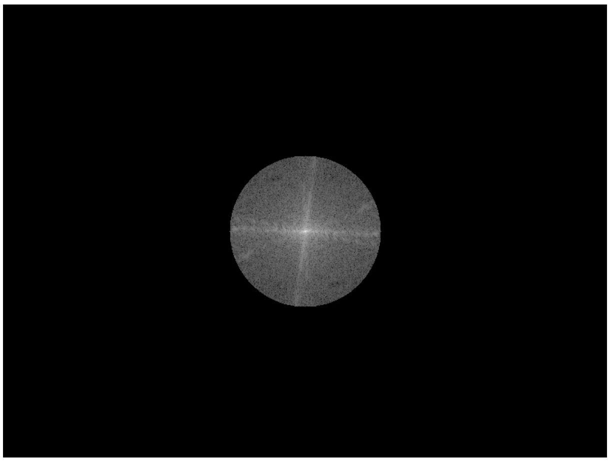
power spectrum $\mathcal{F}^{\text{power}}(f)$:



Low Pass Filters / Example

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



High Pass Filters

High frequencies are responsible for sharp edges,
low frequencies for constant and slowly changing areas.

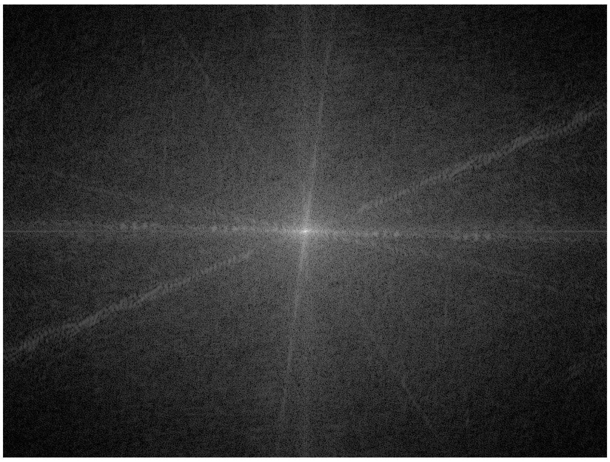
High pass filters

- retain only high frequencies $\omega \geq \omega_{\text{min}}$,
- i.e., filter out low frequencies.
- and thus sharpen an image and detect edges.

High Pass Filters / Example

image f:

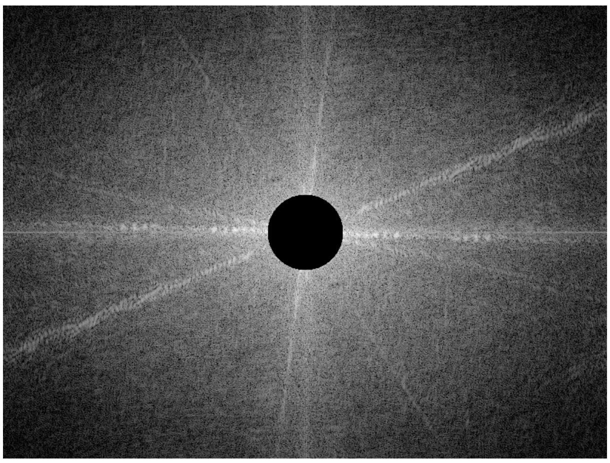
power spectrum $\mathcal{F}^{\text{power}}(f)$:



High Pass Filters / Example

image f:

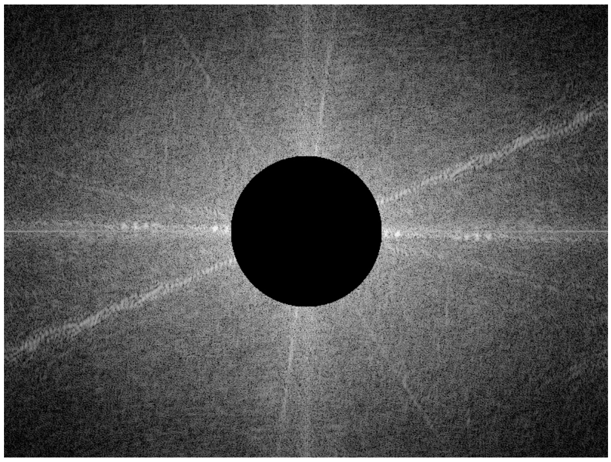
power spectrum $\mathcal{F}^{\text{power}}(f)$:



High Pass Filters / Example

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



Band Pass Filters

High frequencies are responsible for sharp edges,
low frequencies for constant and slowly changing areas.

Band pass filters

– retain only frequencies $\omega \in [\omega_{\text{min}}, \omega_{\text{max}}]$ in a given interval (the **frequency band**),
– i.e., filter out low and high frequencies.
– and thus detect edges.

Band Pass Filters / Example

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:

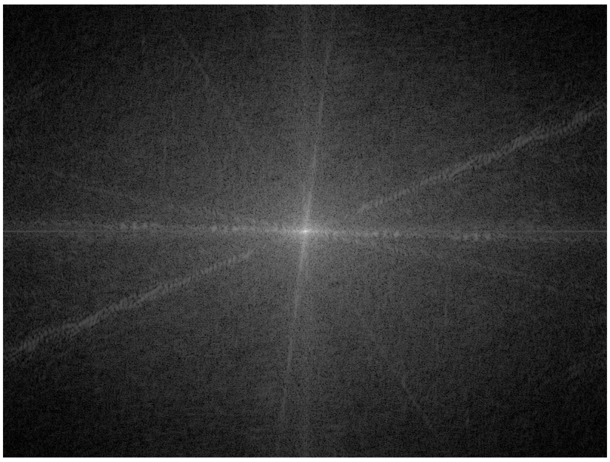
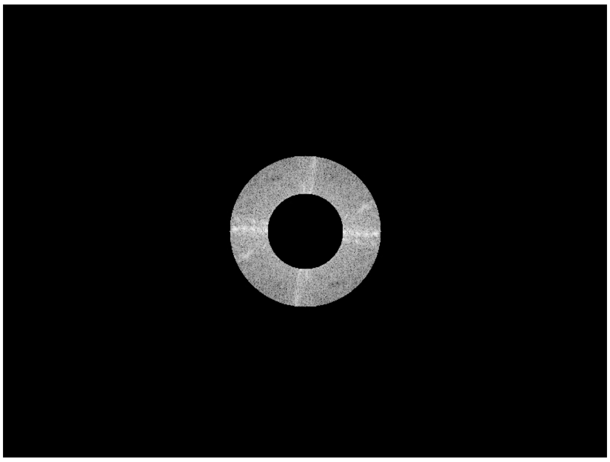


Image Analysis / 6. Applications

Band Pass Filters / Example

image f:

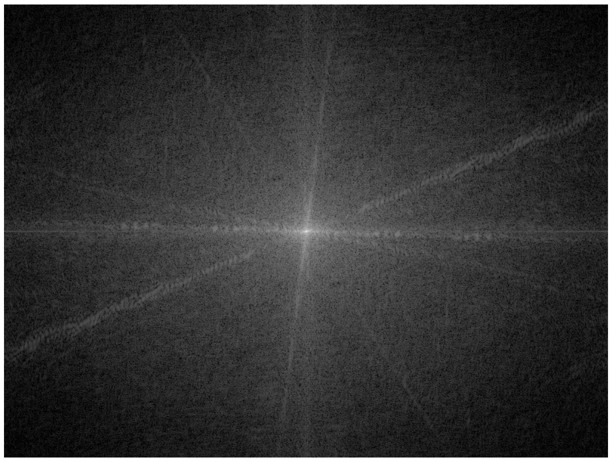
power spectrum $\mathcal{F}^{\text{power}}(f)$:



Reducing Periodic Noise

image f:

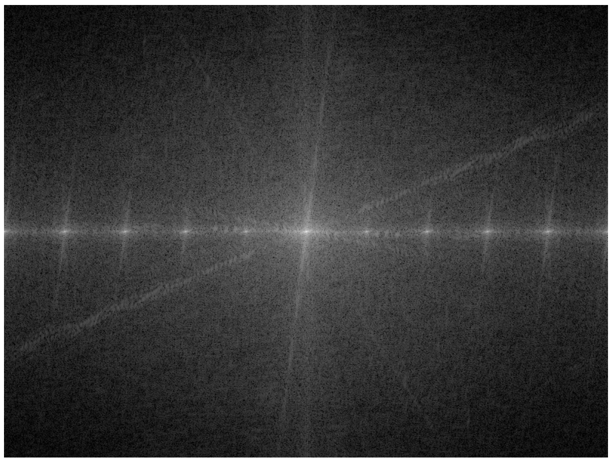
power spectrum $\mathcal{F}^{\text{power}}(f)$:



Reducing Periodic Noise

image f:

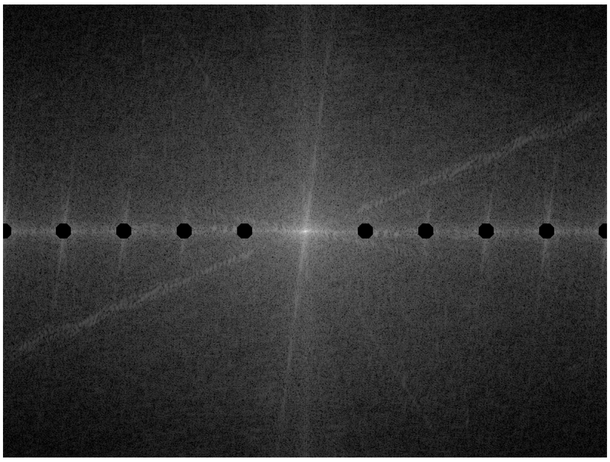
power spectrum $\mathcal{F}^{\text{power}}(f)$:



Reducing Periodic Noise

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



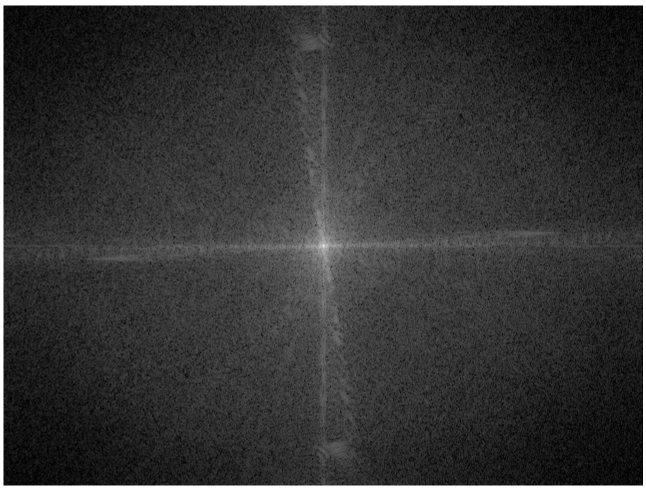
Periodic noise can be reduced by filtering out the frequencies belonging to the periodic noise pattern.

This also can be understood as a simple method for **inpainting**.

Reducing Salt and Pepper Noise

image f:

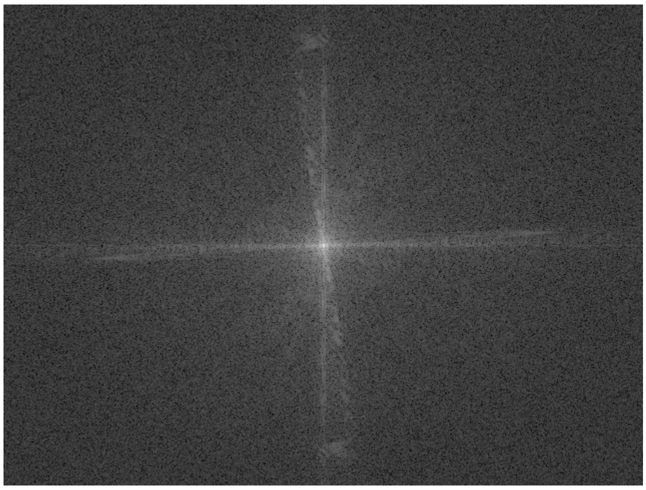
power spectrum $\mathcal{F}^{\text{power}}(f)$:



Reducing Salt and Pepper Noise

image f:

power spectrum $\mathcal{F}^{\text{power}}(f)$:



Reducing non-periodic noise patterns via frequency filters is difficult.

Deconvolution via Fourier Transform

Assume, an image f has been corrupted by a convolution with a kernel k (e.g., blurred):

$$g = k * f$$

If the kernel k is known, one can "undo" the convolution using the Fourier transform:

$$\mathcal{F}(g) = \mathcal{F}(k * f) = \mathcal{F}(k) \cdot \mathcal{F}(f)$$

$$\mathcal{F}(f) = \frac{\mathcal{F}(g)}{\mathcal{F}(k)}$$

$$f = \mathcal{F}^{-1}\mathcal{F}(f) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(g)}{\mathcal{F}(k)}\right)$$

Schedule

Schedule until Christmas:

- next Tue., 9.12., no lecture.
- next Wed., 10.12, 10-12 lecture.
- Tue., 16.12., no lecture.
- Wed., 17.12, 10-12 lecture.