

Image Analysis

4. Wavelets

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Course on Image Analysis, winter term 2011/12

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1. Haar Wavelets

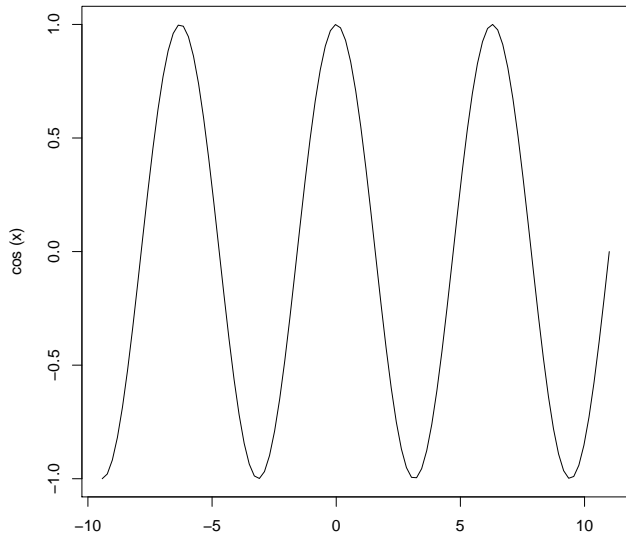
2. Daubechies Wavelets

3. Two-dimensional Wavelets

Basis Functions

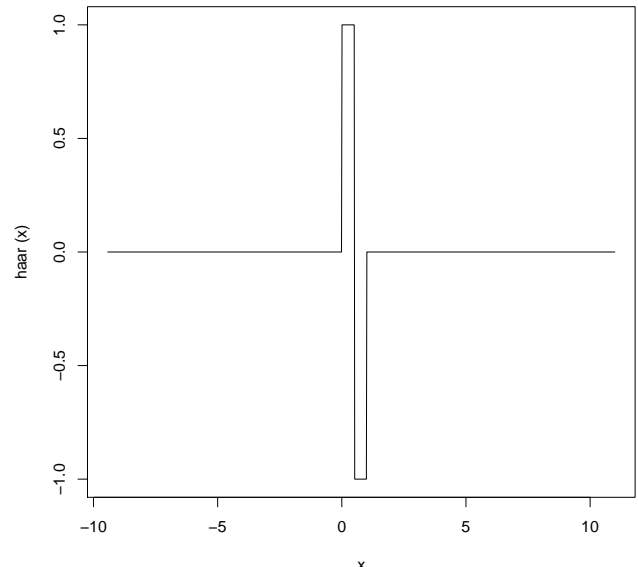
Fourier Analysis:

$$\psi(x) := \cos x$$



Wavelets:

$$\psi(x) := \text{haar}(x) := \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ -1, & x \in [\frac{1}{2}, 1) \\ 0, & \text{else} \end{cases}$$



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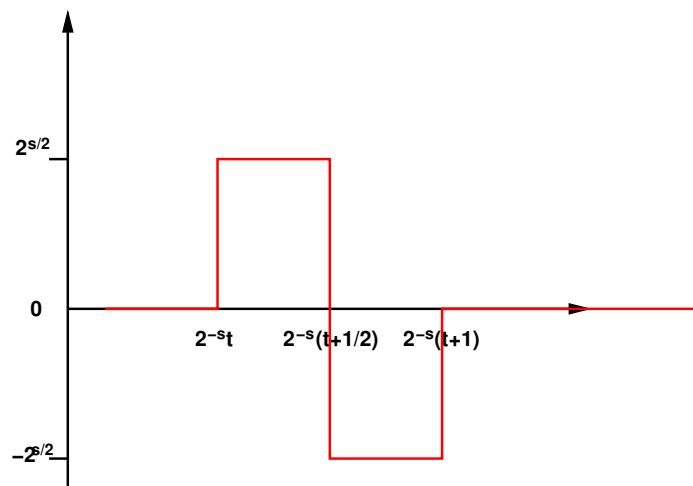
Basis Functions

Fourier Analysis:

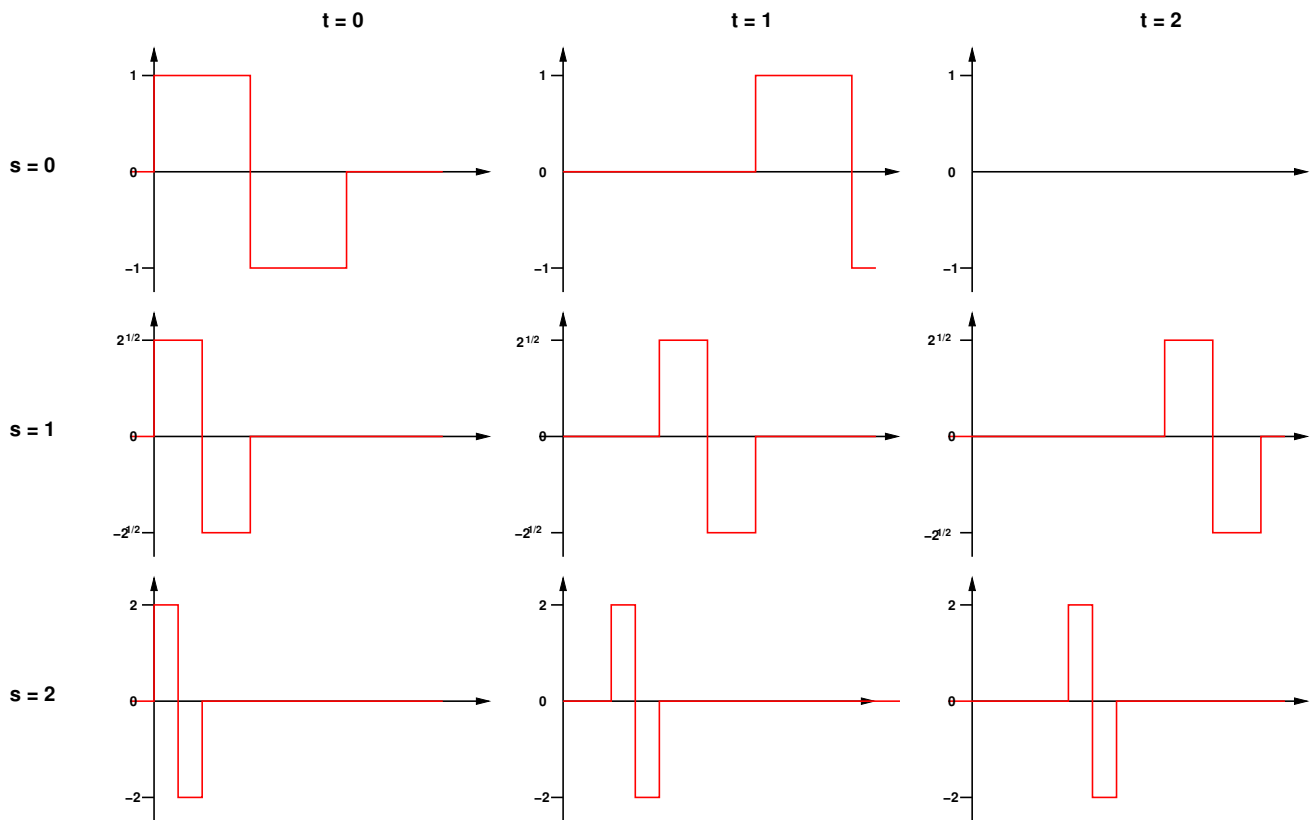
$$\psi_\omega(x) := \cos 2\pi\omega x$$

Wavelets:

$$\begin{aligned} \psi_{s,t}(x) &:= \sqrt{2^s} \cdot \text{haar}(2^s x - t) \\ &= \sqrt{2^s} \cdot \begin{cases} 1, & x \in (2^{-s}t, 2^{-s}(t + \frac{1}{2})) \\ -1, & x \in [2^{-s}(t + \frac{1}{2}), 2^{-s}(t + 1)) \\ 0, & \text{else} \end{cases} \end{aligned}$$



Basis Functions



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Orthogonality of Basis Functions

Obviously, two distinct Haar basis functions $\psi_{s,t}$ and $\psi_{s',t'}$ with $s, t, s', t' \in \mathbb{Z}$ are orthogonal:

$$\langle \psi_{s,t}, \psi_{s',t'} \rangle := \int_{-\infty}^{\infty} \psi_{s,t}(x) \cdot \psi_{s',t'}(x) dx = 0$$

And

$$\langle \psi_{s,t}, \psi_{s,t} \rangle = 1$$

Proof.

If they have the same scale ($s = s'$), then their support does not overlap.

If they have different scale, say $s > s'$, then $\psi_{s,t}$ is constant on the support of $\psi_{s',t'}$, i.e., the integral averages to zero.

$\langle \psi_{s,t}, \psi_{s,t} \rangle$ integrates $\sqrt{2^s} \cdot \sqrt{2^s} = 2^s$ over the support 2^{-s} .

Wavelet Representation

Theorem (Wavelet Representation). Let $\psi_{s,t}$, $s, t \in \mathbb{Z}$ be a set of Wavelet basis functions.

Every function $f : \mathbb{R} \rightarrow \mathbb{R}$ (satisfying some regularity conditions) can be written as

$$f(x) = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{s,t} \psi_{s,t}(x)$$

with coefficients $c_{s,t} \in \mathbb{R}$.

The coefficients $c_{s,t}$ can be computed as follows:

$$c_{s,t} = \int_{-\infty}^{\infty} f(x) \psi_{s,t}(x) dx$$

Haar Wavelet Representation

For the Haar basis functions this yields

$$f(x) = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{s,t} \cdot \sqrt{2^s} \text{haar}(2^s x - t)$$

and

$$c_{s,t} = \sqrt{2^s} \left(\int_{2^{-s}t}^{2^{-s}(t+\frac{1}{2})} f(x) dx - \int_{2^s(t+\frac{1}{2})}^{2^{-s}(t+1)} f(x) dx \right)$$

Haar Wavelets / Computing Coefficients

The values of integrals with a simple rectangle impulse on different scales can be computed recursively:

$$a_{s,t} := \sqrt{2^s} \int_{2^{-s}t}^{2^{-s}(t+1)} f(x) dx$$

$$a_{s,t} = \frac{1}{\sqrt{2}} (a_{s+1,2t} + a_{s+1,2t+1})$$

The coefficients of the Haar wavelet can be computed from these values via

$$c_{s,t} = \frac{1}{\sqrt{2}} (a_{s+1,2t} - a_{s+1,2t+1})$$

Haar Wavelets / Discrete Wavelet Transform

For a finite discrete signal f of length 2^n the function can already be represented by a finite sum of Haar wavelets:

$$f(x) = a_{-n,0} + \sum_{s=-n}^{-1} \sum_{t=0}^{2^{n+s}-1} c_{s,t} \cdot \sqrt{2^s} \text{haar}(2^s x - t)$$

i.e., a composition of Haar wavelets with supports 2, 4, 8 etc.

The initial a values are just the signal values:

$$a_{s=0,t} := \int_{2^{-s}t}^{2^{-s}(t+1)} f(x) dx$$

$$= \int_t^{t+1} f(x) dx$$

$$= \sum_{x=t}^{<t+1} f(x) = f(t)$$

Haar Wavelets / Computing Coefficients / Example

Let

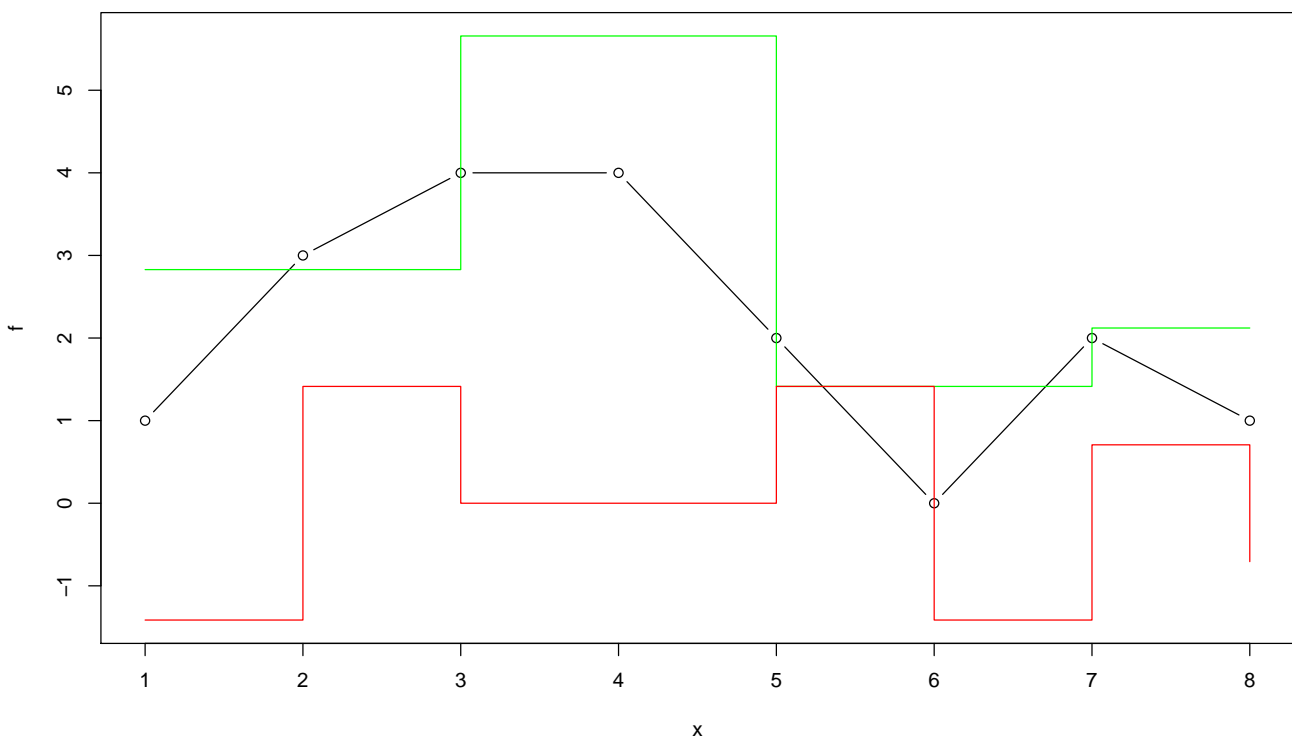
$$f = (1, 3, 4, 4, 2, 0, 2, 1)$$

Then the discrete Haar wavelet transform of f can be computed as follows:

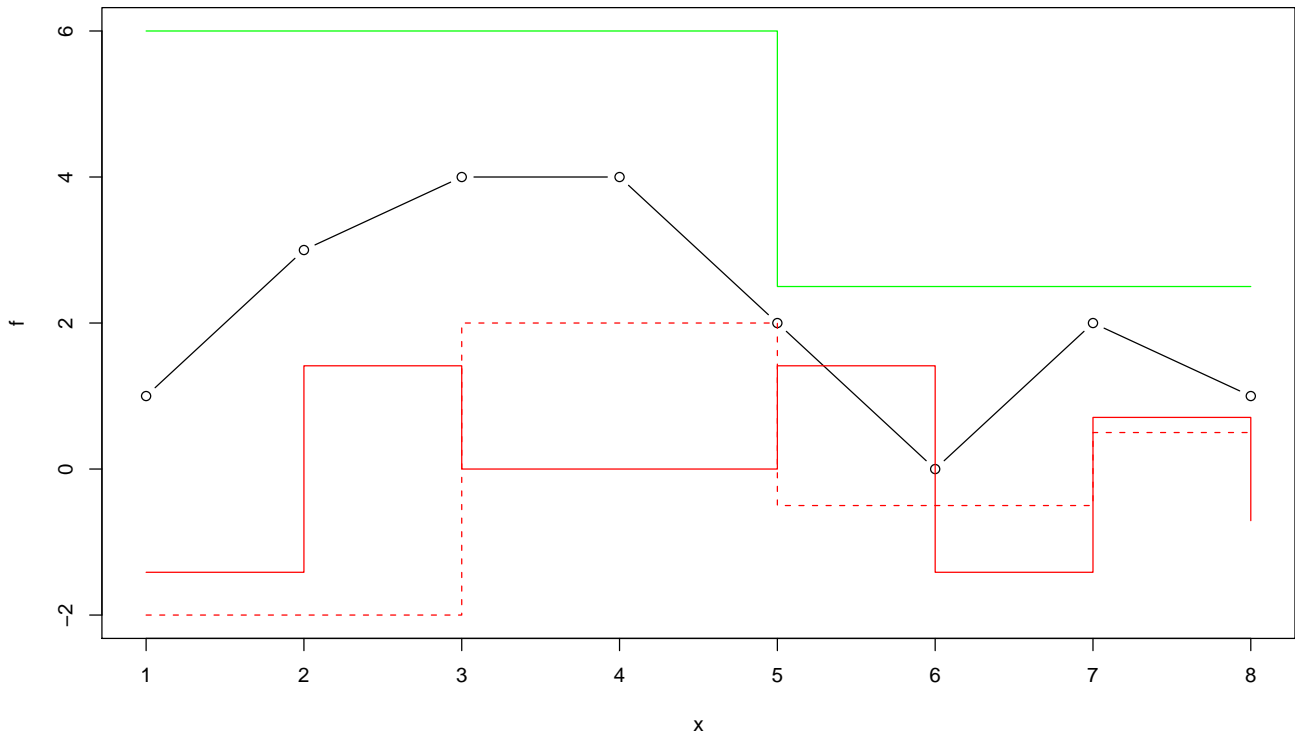
s	t							
	0	1	2	3	4	5	6	7
$a_0 = f$	1	3	4	4	2	0	2	1
a_{-1}	2.83	5.66	1.41	2.12	—	—	—	—
c_{-1}	-1.41	0.00	1.41	0.71	—	—	—	—
a_{-2}	6	2.5	—	—	—	—	—	—
c_{-2}	-2	-0.5	—	—	—	—	—	—
a_{-3}	6.01	—	—	—	—	—	—	—
c_{-3}	2.47	—	—	—	—	—	—	—

$$\text{DWT}_{\text{haar}}(f) = (6.01, 2.47, -2, -0.5, -1.41, 0.00, 1.41, 0.71)$$

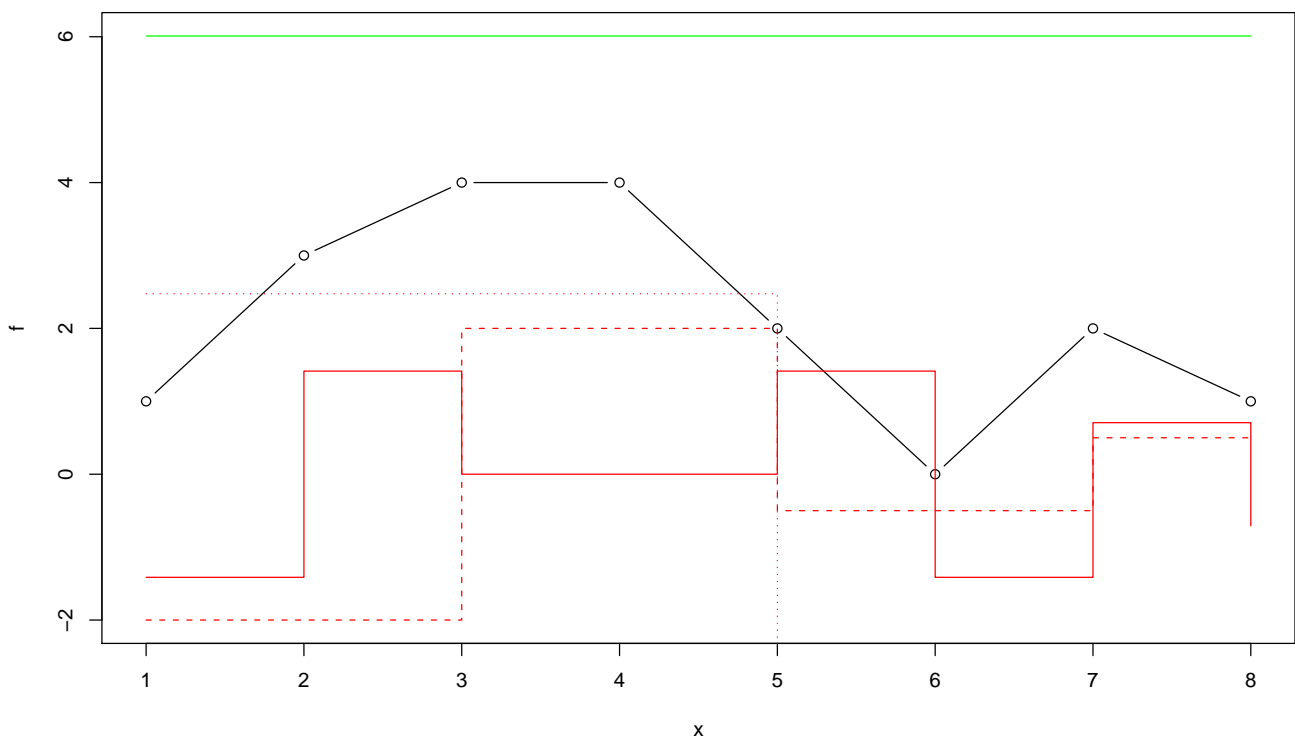
Haar Wavelets / Computing Coefficients / Example



Haar Wavelets / Computing Coefficients / Example



Haar Wavelets / Computing Coefficients / Example



Haar Wavelets / Computing Coefficients

```

1 dwt-haar(sequence  $f = (f(x))_{x=0,\dots,2^n-1}$ ) :
2  $c := (c_{s,t})_{s=0,\dots,n-1; t=0,\dots,2^s-1} := 0$ 
3  $a := (a_{s,t})_{s=0,\dots,n; t=0,\dots,2^s-1} := 0$ 
4  $a_{n,t} := f(t), \quad t = 0, \dots, 2^n - 1$ 
5 for  $s := n - 1, \dots, 0$  do
6   for  $t := 0, \dots, 2^s - 1$  do
7      $a_{s,t} := (a_{s+1,2t} + a_{s+1,2t+1})/\sqrt{2}$ 
8      $c_{s,t} := (a_{s+1,2t} - a_{s+1,2t+1})/\sqrt{2}$ 
9   od
10 od
11 return  $(a_{0,0}, c)$ 

```

Haar Wavelets / Inverse Discrete Wavelet Transform

The DWT easily can be inverted: from

$$a_{s,t} = \frac{1}{\sqrt{2}} (a_{s+1,2t} + a_{s+1,2t+1})$$

$$c_{s,t} = \frac{1}{\sqrt{2}} (a_{s+1,2t} - a_{s+1,2t+1})$$

we get

$$a_{s+1,2t} = \sqrt{2} (a_{s,t} + c_{s,t})/2$$

$$a_{s+1,2t+1} = \sqrt{2} (a_{s,t} - c_{s,t})/2$$

Haar Wavelets / Inverse Discrete Wavelet Transform

```

1 idwt-haar(coefficients  $c = (c_{s,t})_{s=0,\dots,n-1; t=0,\dots,2^s-1}$ ,  $a'$ ) :
2  $a := (a_{s,t})_{s=0,\dots,n; t=0,\dots,2^s-1} := 0$ 
3  $a_{0,0} := a'$ 
4 for  $s := 0, \dots, n - 1$  do
5   for  $t := 0, \dots, 2^s - 1$  do
6      $a_{s+1,2t} := (a_{s,t} + c_{s,t})/\sqrt{2}$ 
7      $a_{s+1,2t+1} := (a_{s,t} - c_{s,t})/\sqrt{2}$ 
8   od
9 od
10  $f := (f(x))_{x=0,\dots,2^n-1} := a_{n,x}, \quad x = 0, \dots, 2^n - 1$ 
11 return  $f$ 

```

1. Haar Wavelets

2. Daubechies Wavelets

3. Two-dimensional Wavelets

Haar Wavelets / Matrix Notation

A single iteration from scale $s + 1$ to s of the discrete Haar wavelet transform can be described by matrix multiplication:

$$\begin{pmatrix} a_0^s \\ c_0^s \\ a_1^s \\ c_1^s \\ \vdots \\ a_{n-1}^s \\ c_{n-1}^s \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & & & & & \\ 1 & -1 & & & & & & \\ & & 1 & 1 & & & & \\ & & 1 & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & 1 \\ & & & & & & 1 & -1 \end{pmatrix} \begin{pmatrix} a_0^{s+1} \\ a_1^{s+1} \\ a_2^{s+1} \\ \vdots \\ a_{n-1}^{s+1} \\ \vdots \\ a_{2n-1}^{s+1} \end{pmatrix}$$

Daubechies Wavelets / Definition

Ingrid Daubechies (*1954) generalized the Haar wavelets to a family of wavelets now called **Daubechies wavelets** D_k :

$$\begin{pmatrix} a_0^s \\ c_0^s \\ a_1^s \\ c_1^s \\ \vdots \\ a_{n-1}^s \\ c_{n-1}^s \end{pmatrix} = \begin{pmatrix} w_0 & w_1 & w_2 & \dots & w_{k-1} & & & \\ w_{k-1} & -w_{k-2} & w_{k-3} & \dots & -w_0 & & & \\ & & w_0 & w_1 & \dots & w_{k-1} & & \\ & & w_{k-1} & -w_{k-2} & \dots & -w_0 & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \dots & \\ w_2 & \dots & w_{k-1} & & & & w_0 & w_1 \\ w_{k-3} & \dots & -w_0 & & & & w_{k-1} & -w_{k-2} \end{pmatrix} \begin{pmatrix} a_0^{s+1} \\ a_1^{s+1} \\ a_2^{s+1} \\ \vdots \\ a_{n-1}^{s+1} \\ \vdots \\ a_{2n-1}^{s+1} \end{pmatrix}$$

The coefficients w_0, w_1, \dots, w_{k-1} are called the **wavelet filter coefficients**.

Daubechies Wavelets / Definition

The matrix D_k should satisfy two conditions:

1. **Orthogonality**, i.e., $D_k D_k^T = 1$:

$$\sum_{i=0}^{k-1} w_i^2 = 1$$

$$\sum_{i=0}^{k-1-2m} w_i w_{2m+i} = 0, \quad m = 1, 2, \dots, k/2 - 1$$

2. **Approximation of order $k/2$** , i.e., the first $k/2$ moments vanish.

For D_4 this means:

$$w_3 - w_2 + w_1 - w_0 = 0$$

$$0w_3 - 1w_2 + 2w_1 - 3w_0 = 0$$

Daubechies Wavelets / Definition

In general, this are k conditions for the k coefficients of D_K leading to a unique solution:

$$w(D_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$w(D_4) = \left(\frac{1 + \sqrt{3}}{4\sqrt{2}}, \frac{3 + \sqrt{3}}{4\sqrt{2}}, \frac{3 - \sqrt{3}}{4\sqrt{2}}, \frac{1 - \sqrt{3}}{4\sqrt{2}} \right)$$

D_2 is the Haar wavelet.

$w(D_6)$ also can be computed analytically; the coefficients of the higher order Daubechies wavelets can only be computed numerically.

Daubechies Wavelets / DWT Algorithm

```

1 dwt-daubechies(sequence  $f = (f(x))_{x=0,\dots,2^n-1}$ ,  $k$ ) :
2  $w := (w(x))_{x=0,\dots,k-1} := \text{getDaubechiesWaveletCoefficients}(k)$ 
3  $c := (c_{s,t})_{s=0,\dots,n-1; t=0,\dots,2^s-1} := 0$ 
4  $a := (a_{s,t})_{s=0,\dots,n; t=0,\dots,2^s-1} := 0$ 
5  $a_{n,t} := f(t)$ ,  $t = 0, \dots, 2^n - 1$ 
6 for  $s := n - 1, \dots, 0$  do
7   for  $t := 0, \dots, 2^s - 1$  do
8      $a_{s,t} := 0$ 
9      $c_{s,t} := 0$ 
10    for  $x := 0, \dots, k - 1$  do
11       $a_{s,t} := a_{s,t} + a_{s+1,2t+x \bmod 2^{s+1}} w(x)$ 
12       $c_{s,t} := c_{s,t} + a_{s+1,2t+x \bmod 2^{s+1}} (-1)^x w(k - 1 - x)$ 
13    od
14  od
15 od
16 return  $(a_{0,0}, c)$ 

```

Generic DWT Algorithm (1/2)

```

1 dwt-generic(sequence  $f = (f(x))_{x=0,\dots,2^n-1}$ , wavelet transform  $W$ ) :
2  $c := (c_{s,t})_{s=0,\dots,n-1; t=0,\dots,2^s-1} := 0$ 
3  $a := (a_{s,t})_{s=0,\dots,n; t=0,\dots,2^s-1} := 0$ 
4  $a_{n,t} := f(t)$ ,  $t = 0, \dots, 2^n - 1$ 
5 for  $s := n - 1, \dots, 0$  do
6    $(c_{s,\cdot}, a_{s,\cdot}) := \text{dwt-iteration}(a_{s+1,\cdot}, W)$ 
7 od
8 return  $(a_{0,0}, c)$ 
9
10 dwt-iteration(sequence  $f = (f(x))_{x=0,\dots,2^{n+1}-1}$ , wavelet transform  $W$ ) :
11  $a := (a_t)_{t=0,\dots,2^n-1} := 0$ 
12  $c := (c_t)_{t=0,\dots,2^n-1} := 0$ 
13 for  $t := 0, \dots, 2^n - 1$  do
14    $(a_t, c_t) := W((f_{2t+x \bmod 2^{n+1}})_{x=0,\dots,2^{n+1}-1})$ 
15 od
16 return  $(c, a)$ 

```

Generic DWT Algorithm (2/2)

```

1 W-haar(sequence  $a = (a(t))_{t=0,\dots,2^n-1}$ ) :
2 return  $((a_0 + a_1)/\sqrt{2}, (a_0 - a_1)/\sqrt{2})$ 
3
4 W-daubechies-k(sequence  $a = (a(t))_{t=0,\dots,2^n-1}$ ) :
5  $w := (w(x))_{x=0,\dots,k-1} := \text{getDaubechiesWaveletCoefficients}(k)$ 
6  $a' := 0$ 
7  $c := 0$ 
8 for  $x := 0, \dots, k - 1$  do
9    $a' := a' + a_x w(x)$ 
10   $c' := c' + a_x (-1)^x w(k - 1 - x)$ 
11 od
12 return  $(a', c')$ 

```

Daubechies Wavelets / Inverse DWT Algorithm

As D_k is orthogonal, one can easily compute the inverse of the DWT via:

$$\begin{pmatrix} a_0^{s+1} \\ a_1^{s+1} \\ a_2^{s+1} \\ \vdots \\ a_{n-1}^{s+1} \\ \vdots \\ a_{2n-1}^{s+1} \end{pmatrix} = \begin{pmatrix} w_0 & w_1 & w_3 & \dots & w_{k-1} \\ w_{k-1} & -w_{k-2} & w_3 & \dots & -w_0 \\ & & w_0 & w_1 & \dots & w_{k-1} \\ & & w_{k-1} & -w_{k-2} & \dots & -w_0 \\ & & & & \dots & \\ & & & & & \dots \\ w_2 & \dots & w_{k-1} & & & w_0 & w_1 \\ w_2 & \dots & -w_{k-1} & & & w_0 & -w_1 \end{pmatrix}^T \begin{pmatrix} a_0^s \\ c_0^s \\ a_1^s \\ c_1^s \\ \vdots \\ a_{n-1}^s \\ c_{n-1}^s \end{pmatrix}$$

Daubechies Wavelets / Inverse DWT Algorithm

```

1 idwt-daubechies(coefficients  $c = (c_{s,t})_{s=0,\dots,n-1; t=0,\dots,2^s-1}$ ,  $a', k$ ) :
2  $w := (w(x))_{x=0,\dots,k-1} := \text{getDaubechiesWaveletCoefficients}(k)$ 
3  $a := (a_{s,t})_{s=0,\dots,n; t=0,\dots,2^s-1} := 0$ 
4  $a_{0,0} := a'$ 
5 for  $s := 0, \dots, n - 1$  do
6   for  $t := 0, \dots, 2^s - 1$  do
7     for  $x := 0, \dots, k - 1$  do
8        $a_{s+1,2t+x \bmod 2^{s+1}} := a_{s+1,2t+x \bmod 2^{s+1}} + a_{s,t+x \bmod 2^s} w(x)$ 
9        $a_{s+1,2t+1+x \bmod 2^{s+1}} := a_{s+1,2t+1+x \bmod 2^{s+1}} + c_{s,t+x \bmod 2^s} (-1)^x w(k - 1 - x)$ 
10    od
11  od
12 od
13  $f := (f(x))_{x=0,\dots,2^n-1} := a_{n,x}, \quad x = 0, \dots, 2^n - 1$ 
14 return  $f$ 

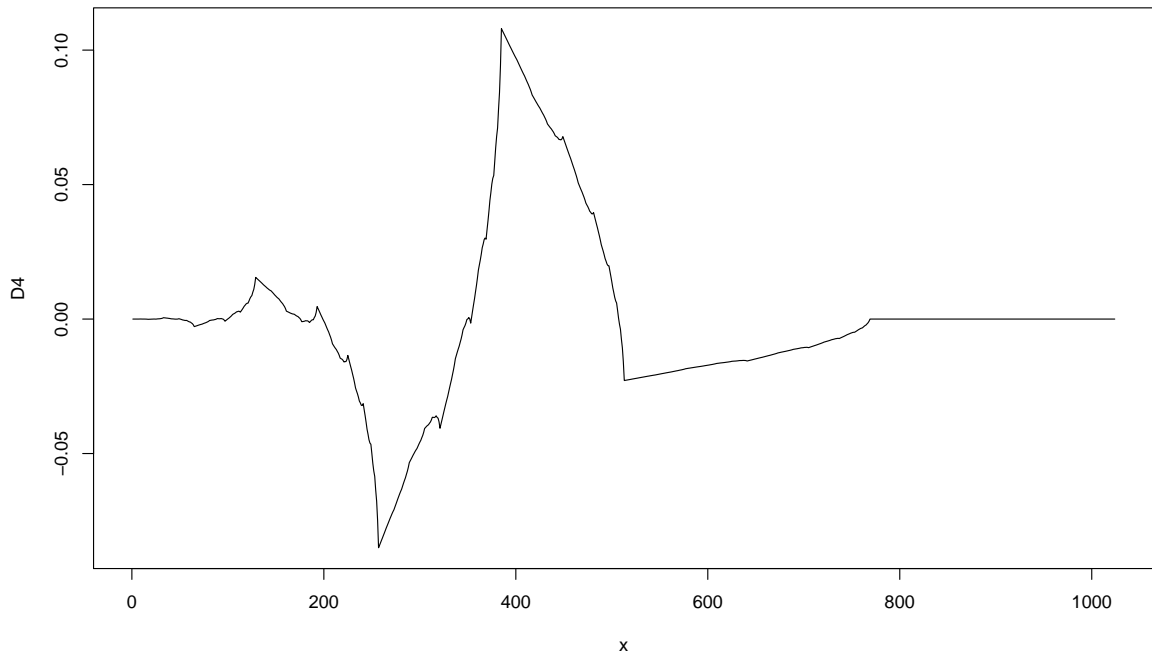
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Daubechies Wavelets / Daubechies Wavelet Basis Functions

Daubechies wavelets have been defined implicitly by their wavelet coefficients in the DWT.

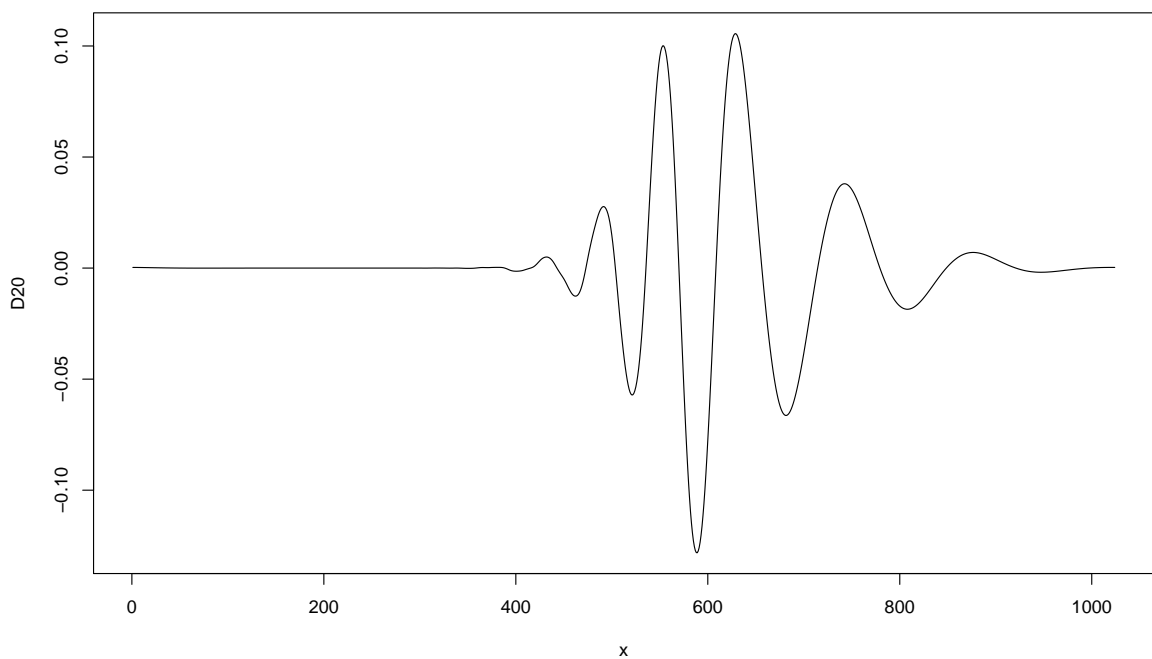
But how does a Daubechies wavelet look like?

We run a unit vector of length 1024 through IDWT, i.e., we set all but one coefficient to zero.

Daubechies Wavelets / D_4 Wavelet Basis Function ($s = 8, t = 3$)

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Daubechies Wavelets / D_{20} Wavelet Basis Function ($s = 6, t = 3$)

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2. Daubechies Wavelets

3. Two-dimensional Wavelets

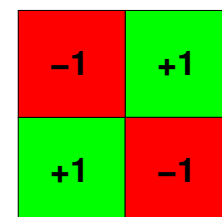
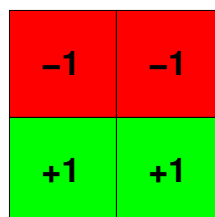
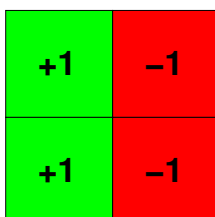
Two-dimensional Haar **Mother Wavelets**

$$\text{haar}^b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad b \in \{1, 2, 3\}$$

$$\text{haar}^1(x, y) := \begin{cases} +1, & \text{if } x \in [0, \frac{1}{2}) \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \\ 0, & \text{else} \end{cases}$$

$$\text{haar}^2(x, y) := \begin{cases} +1, & \text{if } y \in [0, \frac{1}{2}) \\ -1, & \text{if } y \in [\frac{1}{2}, 1) \\ 0, & \text{else} \end{cases}$$

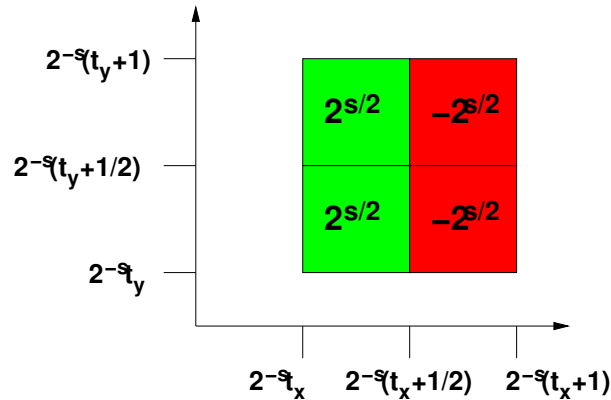
$$\text{haar}^3(x, y) := \begin{cases} +1, & \text{if } (x, y) \in [0, \frac{1}{2})^2 \\ & \cup [\frac{1}{2}, 1)^2 \\ -1, & \text{if } (x, y) \in [0, \frac{1}{2}) \times [\frac{1}{2}, 1) \\ & \cup [\frac{1}{2}, 1) \times [0, \frac{1}{2}) \\ 0, & \text{else} \end{cases}$$



Two-dimensional Haar Basis Functions

The scaled and translated mother wavelets form a family of two-dimensional Haar basis functions:

$$\psi_{s,t_x,t_y}^b(x) := 2^s \cdot \text{haar}^b(2^s x - t_x, 2^s y - t_y), \quad b \in \{1, 2, 3\}$$



Orthogonality of Haar Basis Functions

Obviously, two distinct Haar basis functions

$$\psi_{s,t_x,t_y}^b \quad \text{and} \quad \psi_{s',t'_x,t'_y}^{b'}$$

with $s, t_x, t_y, s', t'_x, t'_y \in \mathbb{Z}$, $b, b' \in \{1, 2, 3\}$ are orthogonal:

$$\langle \psi_{s,t_x,t_y}^b, \psi_{s',t'_x,t'_y}^{b'} \rangle := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{s,t_x,t_y}^b(x) \cdot \psi_{s',t'_x,t'_y}^{b'}(x) dx dy = 0$$

And

$$\langle \psi_{s,t_x,t_y}^b, \psi_{s,t_x,t_y}^b \rangle = 1$$

Proof. Analogously to the one-dimensional case.

2D Haar Wavelets / Discrete Wavelet Transform

A finite discrete signal f of size $2^n \times 2^n$ can be represented by a finite sum of 2-dimensional Haar wavelets:

$$f(x) = a_{-n,0,0} + \sum_{b=1}^3 \sum_{s=-n}^{-1} \sum_{t_x=0}^{2^{n+s}-1} \sum_{t_y=0}^{2^{n+s}-1} c_{s,t_x,t_y}^b \cdot 2^s \text{haar}^b(2^s x - t_x, 2^s y - t_y)$$

The initial a values are just the signal values:

$$\begin{aligned} a_{s=0,t_x,t_y} &:= \int_{2^{-s} t_x}^{2^{-s}(t_x+1)} \int_{2^{-s} t_y}^{2^{-s}(t_y+1)} f(x, y) dx dy \\ &= \int_{t_x}^{t_x+1} \int_{t_y}^{t_y+1} f(x, y) dx dy \\ &= \sum_{x=t_x}^{<t_x+1} \sum_{y=t_y}^{<t_y+1} f(x, y) = f(t_x, t_y) \end{aligned}$$

2D Haar Wavelets / Computing Coefficients

The values of integrals with a simple rectangle impulse on different scales can be computed recursively:

$$\begin{aligned} a_{s,t_x,t_y} &:= 2^s \int_{2^{-s} t_x}^{2^{-s}(t_x+1)} \int_{2^{-s} t_y}^{2^{-s}(t_y+1)} f(x, y) dx dy \\ a_{s,t_x,t_y} &= \frac{1}{2} (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y} + a_{s+1,2t_x,2t_y+1} + a_{s+1,2t_x+1,2t_y+1}) \end{aligned}$$

The coefficients of the Haar wavelet can be computed from these values via

$$\begin{aligned} c_{s,t_x,t_y}^1 &= \frac{1}{2} (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y} - a_{s+1,2t_x+1,2t_y+1}) \\ c_{s,t_x,t_y}^2 &= \frac{1}{2} (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y} - a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y+1}) \\ c_{s,t_x,t_y}^3 &= \frac{1}{2} (a_{s+1,2t_x,2t_y} + a_{s+1,2t_x+1,2t_y+1} - a_{s+1,2t_x,2t_y+1} - a_{s+1,2t_x+1,2t_y}) \end{aligned}$$

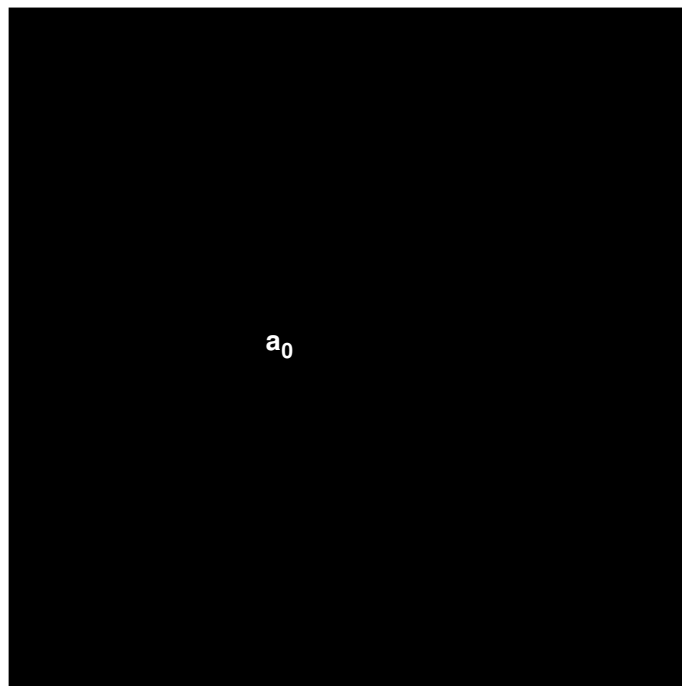
Haar Wavelets / Computing Coefficients

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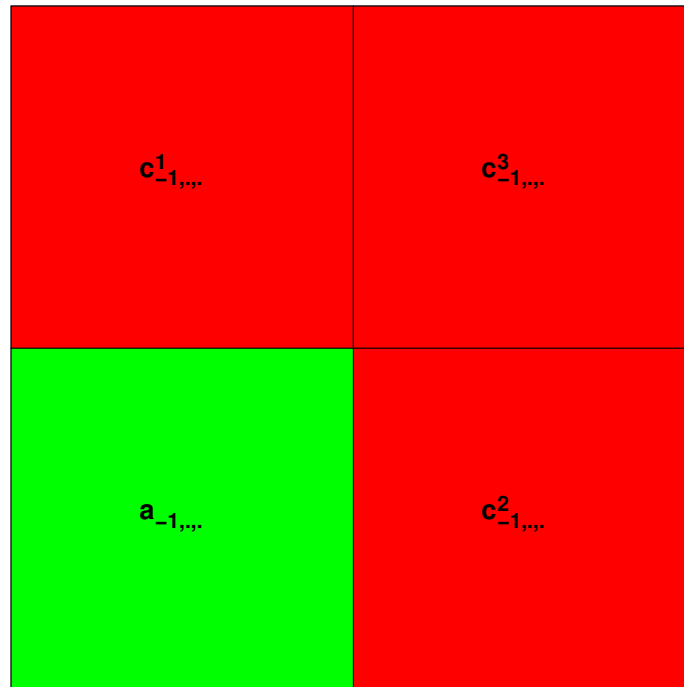
1 dwt2d-haar(image  $f = (f(x, y))_{x=0, \dots, 2^n-1, y=0, \dots, 2^n-1}$ ) :
2  $c := (c_{s, t_x, t_y}^b)_{s=0, \dots, n-1; t_x=0, \dots, 2^s-1; t_y=0, \dots, 2^s-1; b \in \{1, 2, 3\}} := 0$ 
3  $a := (a_{s, t_x, t_y})_{s=0, \dots, n; t_x=0, \dots, 2^s-1; t_y=0, \dots, 2^s-1} := 0$ 
4  $a_{n, t_x, t_y} := f(t_x, t_y), \quad t_x = 0, \dots, 2^n - 1, t_y = 0, \dots, 2^n - 1$ 
5 for  $s := n - 1, \dots, 0$  do
6   for  $t_x := 0, \dots, 2^s - 1$  do
7     for  $t_y := 0, \dots, 2^s - 1$  do
8        $a_{s, t_x, t_y} := (a_{s+1, 2t_x, 2t_y} + a_{s+1, 2t_x+1, 2t_y} + a_{s+1, 2t_x, 2t_y+1} + a_{s+1, 2t_x+1, 2t_y+1})/2$ 
9        $c_{s, t_x, t_y}^1 := (a_{s+1, 2t_x, 2t_y} + a_{s+1, 2t_x, 2t_y+1} - a_{s+1, 2t_x+1, 2t_y} - a_{s+1, 2t_x+1, 2t_y+1})/2$ 
10       $c_{s, t_x, t_y}^2 := (a_{s+1, 2t_x, 2t_y} + a_{s+1, 2t_x+1, 2t_y} - a_{s+1, 2t_x, 2t_y+1} - a_{s+1, 2t_x+1, 2t_y+1})/2$ 
11       $c_{s, t_x, t_y}^3 := (a_{s+1, 2t_x, 2t_y} + a_{s+1, 2t_x+1, 2t_y+1} - a_{s+1, 2t_x, 2t_y+1} - a_{s+1, 2t_x+1, 2t_y})/2$ 
12    od
13  od
14 od
15 return  $(a_{0,0,0}, c)$ 

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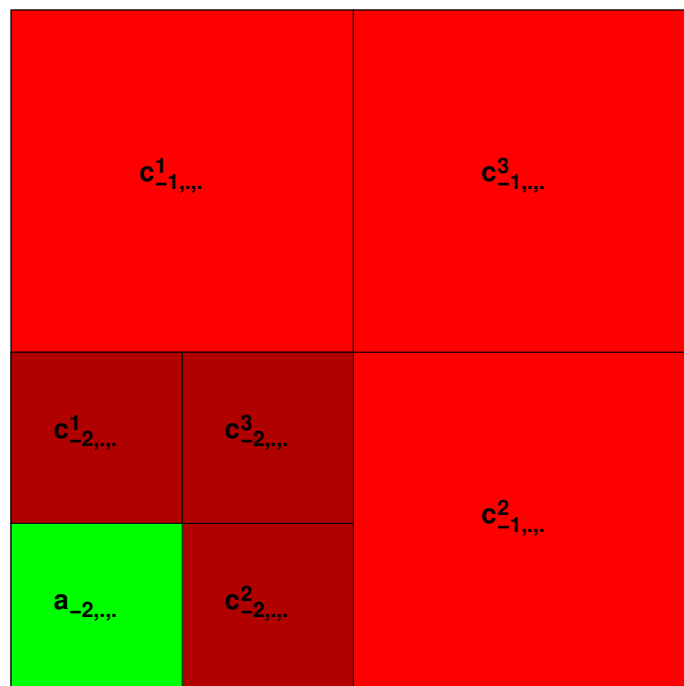
Displaying 2D DWTs



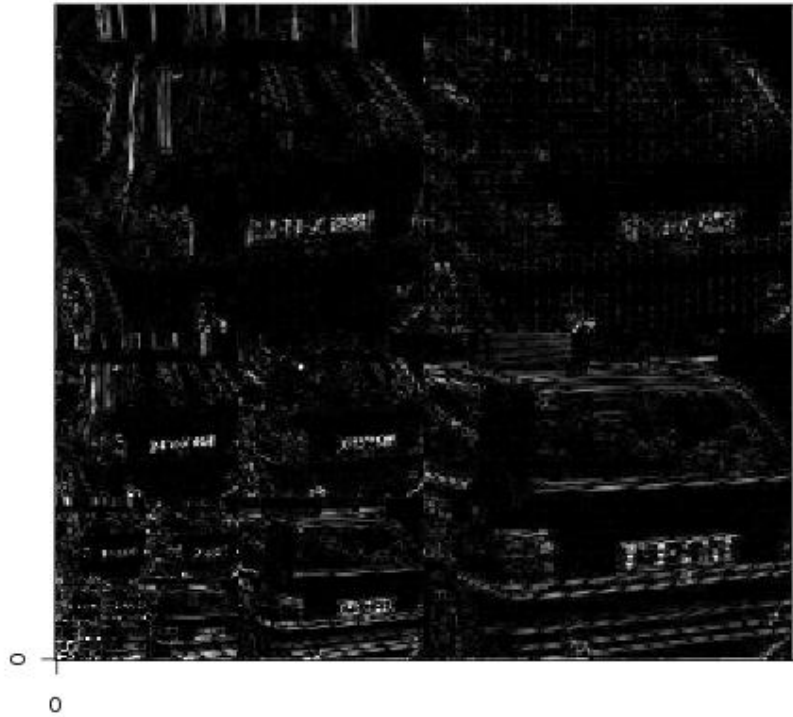
Displaying 2D DWTs



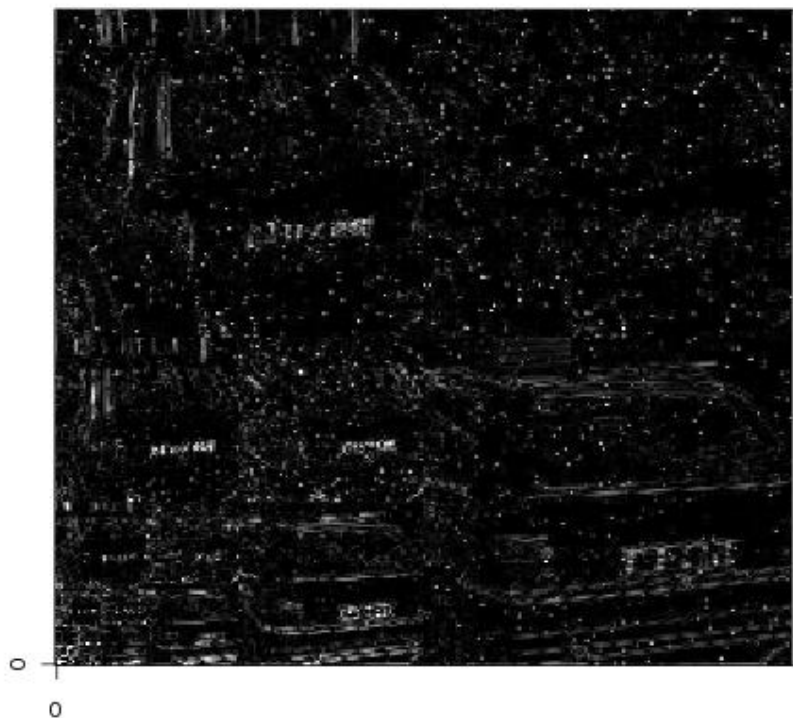
Displaying 2D DWTs



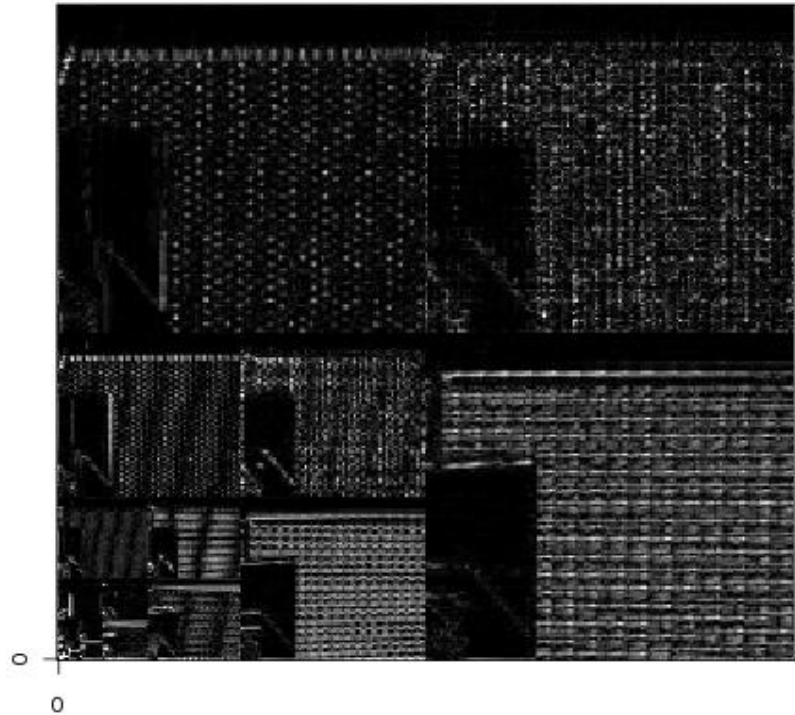
Example



Example



Another Example



Separable 2D Wavelets Bases / Scaling Function

Many 2D wavelet bases can be constructed from 1D wavelet bases and a suitable **scaling function** ϕ (also called **father wavelet**).

For the Haar wavelets the scaling function is just the rectangle impulse:

$$\phi(x) := \begin{cases} 1, & \text{if } x \in [0, 1) \\ 0, & \text{else} \end{cases}$$

In the same manner as for the wavelet functions, one defines scaled and translated variants:

$$\phi_{s,t}(x) := 2^s \phi(2^s x - t)$$

Separable 2D Wavelets Bases / 2D Haar Basis

Obviously, the Haar basis wavelets can be constructed via

$$\psi^1(x, y) = \psi(x) \phi(y)$$

$$\psi^2(x, y) = \phi(x) \psi(y)$$

$$\psi^3(x, y) = \psi(x) \psi(y)$$

and

$$\phi(x, y) = \phi(x) \phi(y)$$

is a suitable 2D scaling function.

Separable wavelet bases allow a generic DWT that

1. applies a 1D DWT to each row of the image and then
2. applies another 1D DWT to each column of the result.

Generic DWT Algorithm

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1 dwt2d-generic(image  $f = (f(x, y))_{x=0, \dots, 2^n-1, y=0, \dots, 2^n-1}$ , wavelet transform  $W$ ) :
2  $c := (c_{s, t_x, t_y}^b)_{s=0, \dots, n-1; t_x=0, \dots, 2^s-1; t_y=0, \dots, 2^s-1; b \in \{1, 2, 3\}} := 0$ 
3  $a := (a_{s, t_x, t_y})_{s=0, \dots, n; t_x=0, \dots, 2^s-1; t_y=0, \dots, 2^s-1} := 0$ 
4  $a_{n, t_x, t_y} := f(t_x, t_y), \quad t_x = 0, \dots, 2^n - 1, t_y = 0, \dots, 2^n - 1$ 
5 for  $s := n - 1, \dots, 0$  do
6    $a' := (a'_{t_x, t_y})_{t_x=0, \dots, 2^{s+1}-1; t_y=0, \dots, 2^s-1} := 0$ 
7    $c' := (c'_{t_x, t_y})_{t_x=0, \dots, 2^{s+1}-1; t_y=0, \dots, 2^s-1} := 0$ 
8   for  $t_x := 0, \dots, 2^{s+1} - 1$  do
9      $(a'_{t_x, \cdot}, c'_{t_x, \cdot}) := \text{dwt-iteration}(a_{s+1, t_x, \cdot}, W)$ 
10  od
11  for  $t_y := 0, \dots, 2^s - 1$  do
12     $(a_{s, \cdot, t_y}, c_{s, \cdot, t_y}^2) := \text{dwt-iteration}(a'_{\cdot, t_y}, W)$ 
13     $(c_{s, \cdot, t_y}^1, c_{s, \cdot, t_y}^3) := \text{dwt-iteration}(c'_{\cdot, t_y}, W)$ 
14  od
15 od
16 od
17 return  $(a_{0,0,0}, c)$ 

```