Machine Learning

A. Supervised Learning
A.4. High-Dimensional Data

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Outline

1. Variable Interactions and Polynomial Models
2. Variable Selection via Forward and Backward Search
3. Minimizing a Function via Coordinate Descent
4. L1 Regularization / The Lasso

High-Dimensional Data

High-dimensional data occurs in different situations:

1. Data that comes naturally with many predictors.
   - e.g., text classification
     (# predictors = # words in the bag-of-words representation, e.g., 30.000)

2. Models that extract many predictor variables from objects to classify.
   - variable interactions
   - derived variables
   - complex objects such as graphs, texts, etc.
     - Situation 1 often really is a special case of this one.

3. Data with few examples compared to the number of variables ("small n, large p").
   - gene expression / microarray data

Outline

Need for higher orders

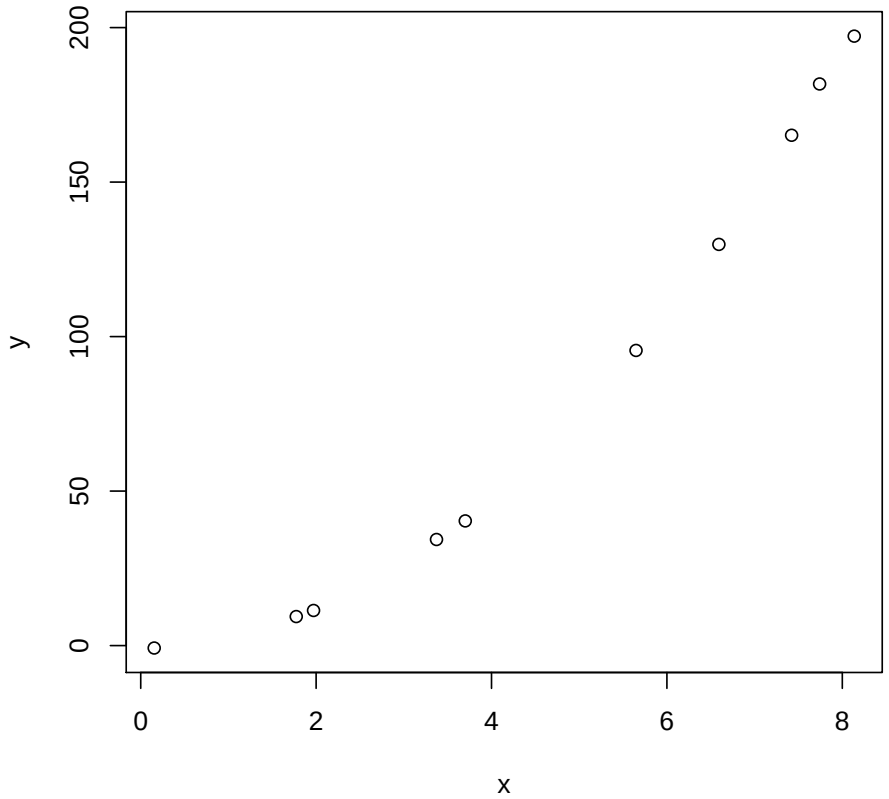
Assume a target variable does not depend linearly on a predictor variable, but say quadratic.

Example: way length vs. duration of a moving object with constant acceleration a.

$$s(t) = \frac{1}{2}at^2 + \epsilon$$

Can we catch such a dependency?

Can we catch it with a linear model?



Need for general transformations

To describe many phenomena, even more complex functions of the input variables are needed.

Example: the number of cells n vs. duration of growth t:

$$n = \beta e^{\alpha t} + \epsilon$$

n does not depend on t directly, but on $e^{\alpha t}$ (with a known α).

Need for variable interactions

In a linear model with two predictors

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$$

Y depends on both, X_1 and X_2.

But changes in X_1 will affect Y the same way, regardless of X_2.

Consider the way length s of a moving object with velocity v and duration t: the way length s of a moving object vs. its constant velocity v and duration t:

$$s = vt + \epsilon$$

- additional $1s$ duration will change the way length not in a uniform way
- high impact for large velocities
- little impact for small velocities

v and t are said to interact: s does not depend only on each predictor separately, but also on their product.

Derived variables

All these cases can be handled by looking at **derived variables**, i.e., instead of

$$Y = \beta_0 + \beta_1 X_1^2 + \epsilon$$
$$Y = \beta_0 + \beta_1 e^{\alpha X_1} + \epsilon$$
$$Y = \beta_0 + \beta_1 X_1 \cdot X_2 + \epsilon$$

one looks at

$$Y = \beta_0 + \beta_1 X_1' + \epsilon$$

with

$$X_1' := X_1^2$$
$$X_1' := e^{\alpha X_1}$$
$$X_1' := X_1 \cdot X_2$$

Derived variables are computed before the fitting process and taken into account either additional to the original variables or instead of.

Polynomial Models

Polynomial models of degree d take into account systematically all interactions of d different variables (including powers up to degree d):

$$\hat{y}(x) := \hat{\beta}_0 + \sum_{m=1}^{M} \hat{\beta}_m x_m$$

degree 1

Polynomial Models

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$$\hat{y}(x) := \hat{\beta}_0 + \sum_{m=1}^{M} \hat{\beta}_m x_m \qquad \text{degree 1}$$

$$\hat{y}(x) := \hat{\beta}_0 + \sum_{m=1}^{M} \hat{\beta}_m x_m + \sum_{m=1}^{M} \sum_{l=m}^{M} \hat{\beta}_{m,l} x_m x_l \qquad \text{degree 2}$$

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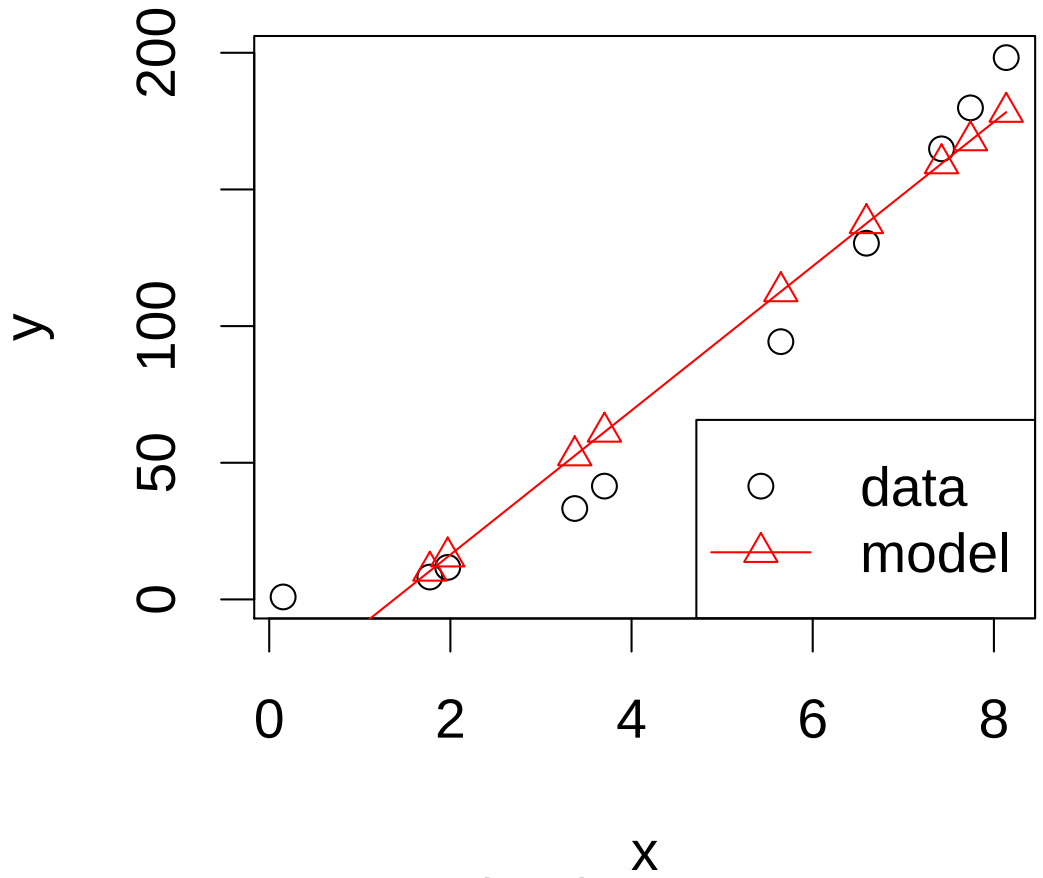
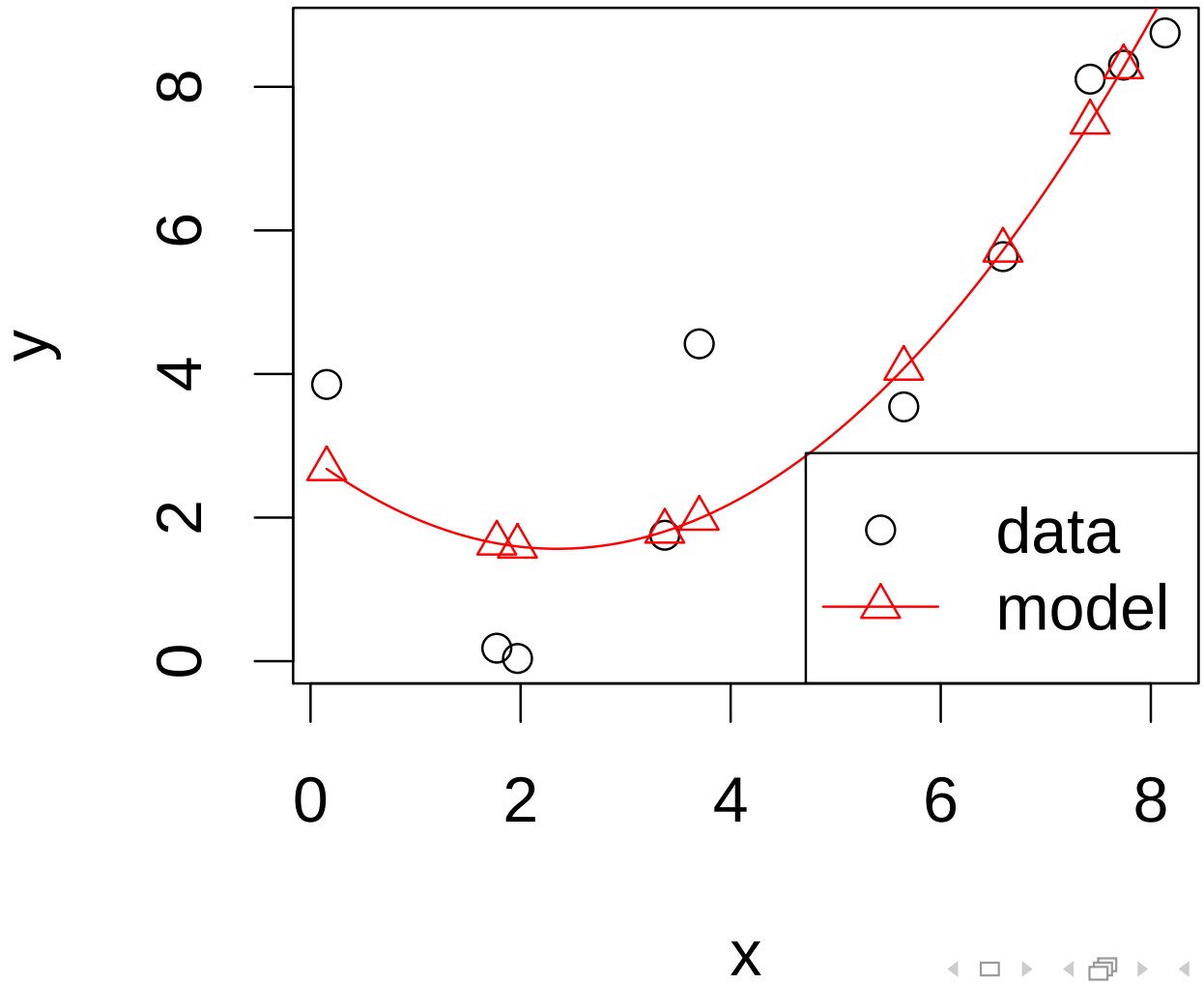
$$+ \sum_{m_1=1}^{M} \sum_{m_2=m_1}^{M} \cdots \sum_{m_d=m_{d-1}}^{M} \hat{\theta}_{m_1,m_2,\ldots,m_d} x_{m_1} x_{m_2} \cdots x_{m_d} \qquad \text{degree } d$$

High Polynomial Degress, High Model Complexity



If a model does not well explain the data,
e.g., if the true model is quadratic, but we try to fit a linear model,
one says, the model **underfits**.

High Polynomial Degress, High Model Complexity



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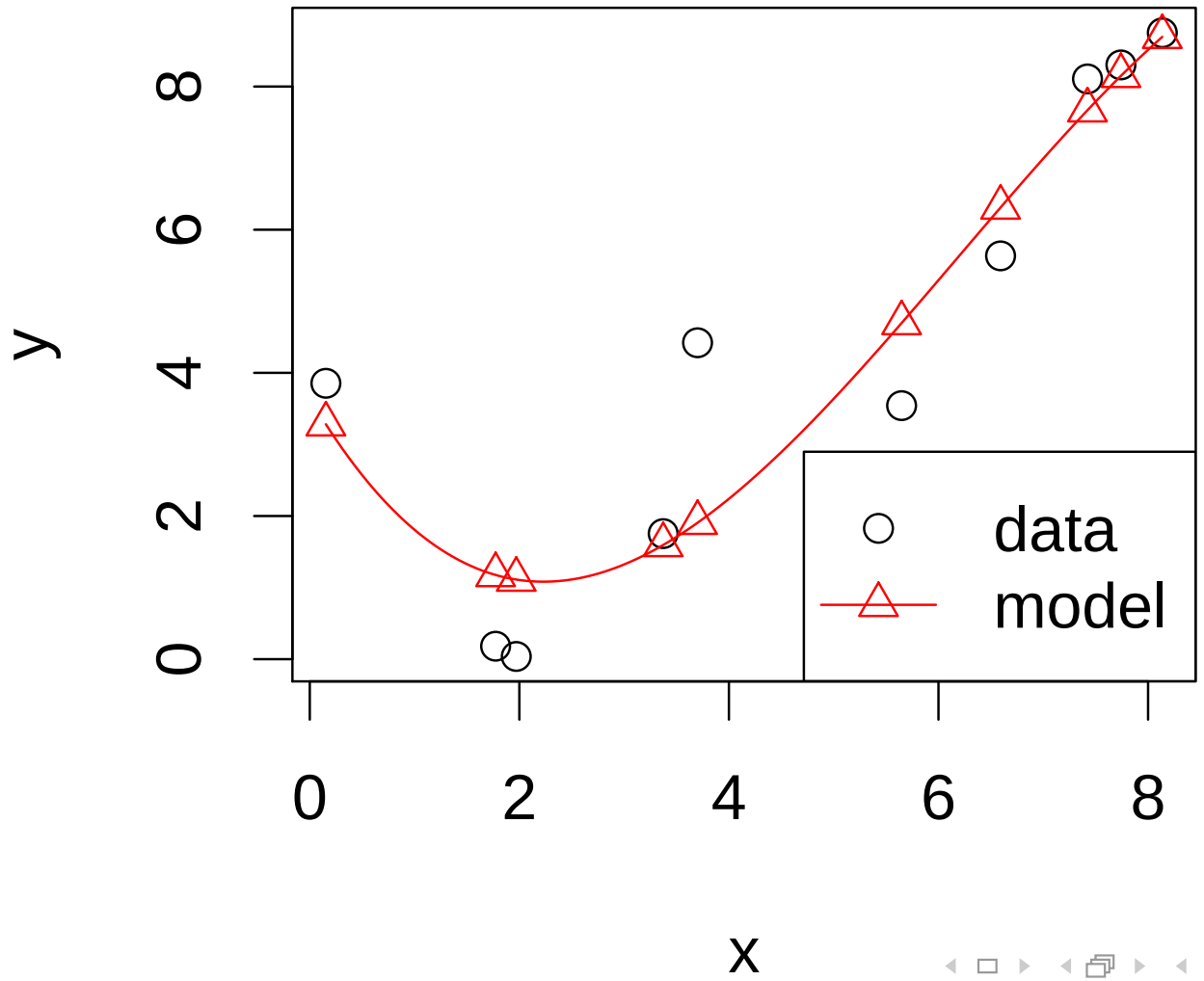

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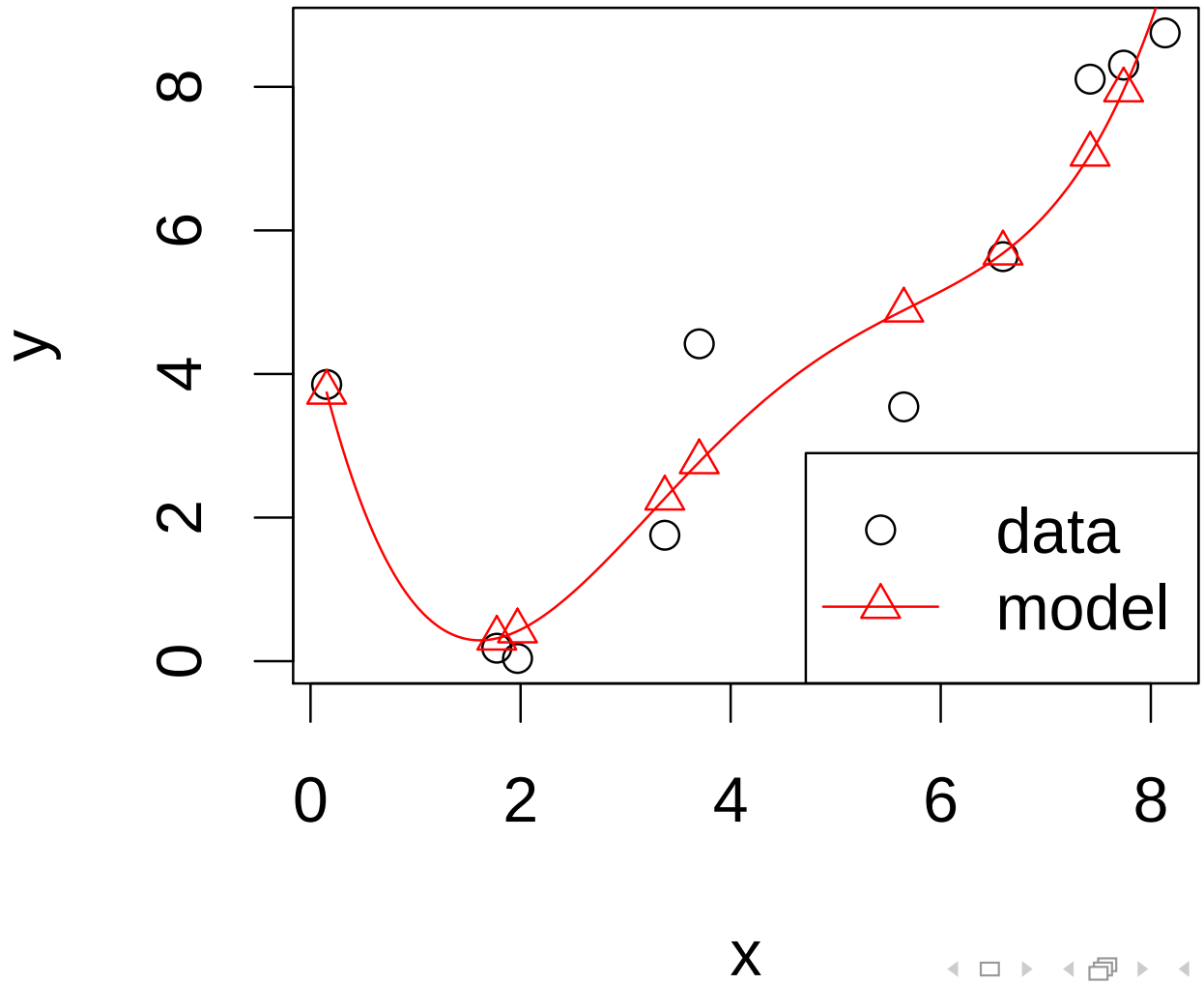
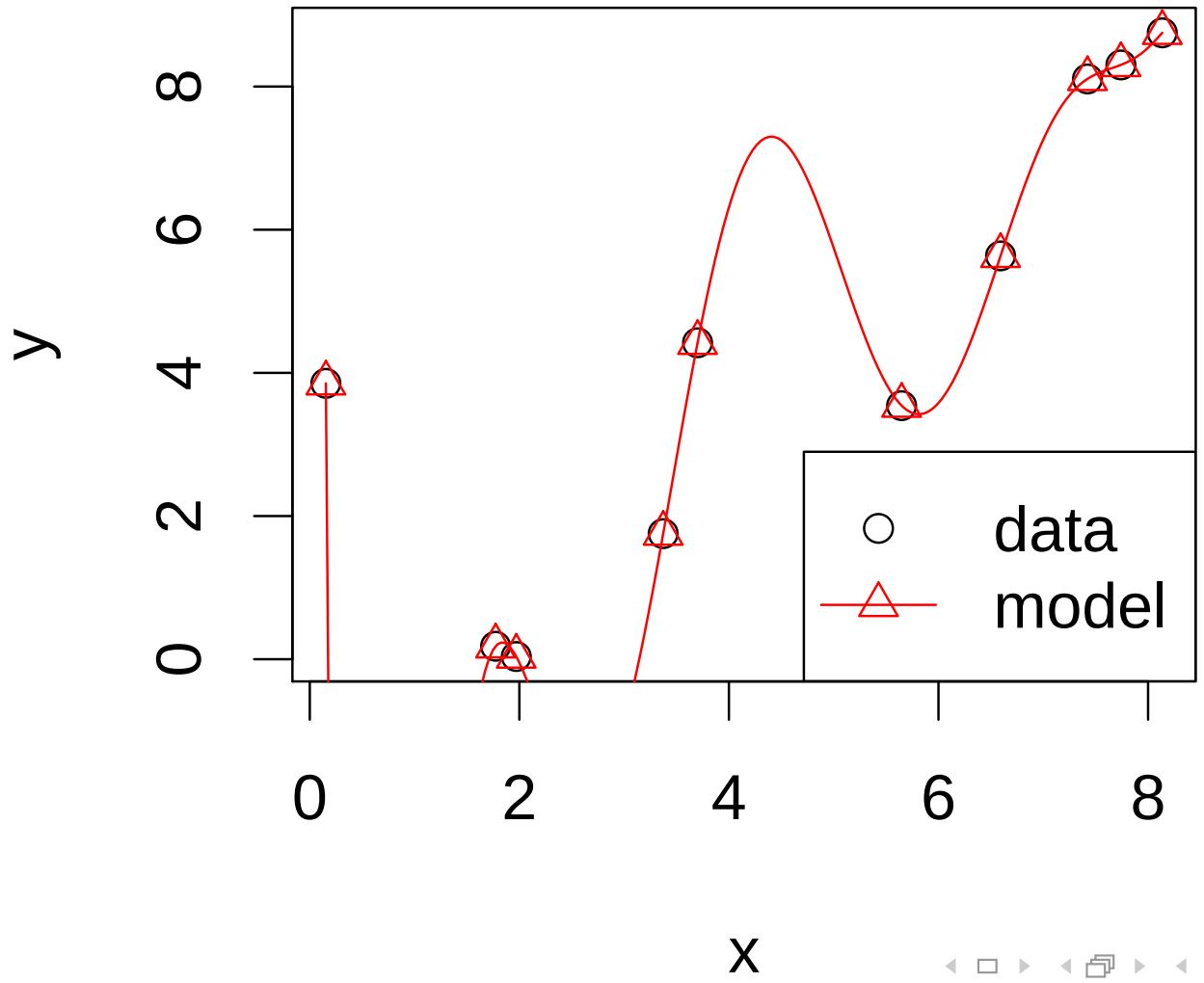


High Polynomial Degress, High Model Complexity



High Polynomial Degress, High Model Complexity

If to data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

consisting of n points we fit

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_{n-1} x^{n-1} \tag{1}$$

i.e., a polynomial with degree $n-1$, then this results in an interpolation of the data points
(if there are no repeated measurements, i.e., points with the same value of x.)

As the polynomial

$$\hat{y}(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

is of this type, and has minimal RSS $= 0$.

Outline

The Variable Selection Problem

Given a data set $\mathcal{D}^{\text{train}} \subseteq \mathbb{R}^M \times \mathcal{Y}$,
an error measure err,
a model class with a learning algorithm $\mathcal{A}$,

find the subset $V \subseteq \{1, 2, \ldots, M\}$ of (relevant) variables s.t. the model

$$\hat{y} := \mathcal{A}(\pi_V(\mathcal{D}^{\text{train}}))$$

learned on this subset V is best, i.e., for new test data $\mathcal{D}^{\text{test}}$ it's test error

$$\text{err}(\hat{y}, \mathcal{D}^{\text{test}}),$$

is minimal.

Projection onto predictors V:

$$\pi_V(x, y) := (x_{i_1}, x_{i_2}, \ldots, x_{i_{\tilde{M}}}, y), \quad \text{for } V := \{i_1, i_2, \ldots, i_{\tilde{M}}\}$$

Greedy Search

- All 2^M subsets are too many to test (for larger M).
- Use a simple greedy search.
- **forward search:**
  - start with no variables.
  - test adding one more variable not yet in the model.
  - add the one leading to lowest validation error.
- **backward search:**
  - start with all variables.
  - test removing one more variable still in the model.
  - remove the one leading to lowest validation error.
- Does not guarantee to find the best variables subset.
  (But usually finds a useful one.)

Forward Search

procedure SELECTVARS-FORWARD($\mathcal{D}^{\text{train}\prime} \subseteq \mathbb{R}^M \times \mathcal{Y}, \text{err}, \mathcal{A}$)
  $(\mathcal{D}^{\text{train}}, \mathcal{D}^{\text{val}}) := \text{split}(\mathcal{D}^{\text{train}\prime})$
  $V := \emptyset$
  $e_{\text{allbest}} := \text{err}(\mathcal{A}(\pi_V(\mathcal{D}^{\text{train}})), \pi_V(\mathcal{D}^{\text{val}}))$
  $v_{\text{best}} := 1$
  while $v_{\text{best}} \neq 0$ **do**
    $v_{\text{best}} := 0$
    $e_{\text{best}} := e_{\text{allbest}}$
    for $v \in \{1, 2, \ldots, M\} \setminus V$ **do**
      $V' := V \cup \{v\}$
      $\hat{y} := \mathcal{A}(\pi_{V'}(\mathcal{D}^{\text{train}}))$
      $e := \text{err}(\hat{y}, \pi_{V'}(\mathcal{D}^{\text{val}}))$
      if $e < e_{\text{best}}$ **then**
        $v_{\text{best}} := v$
        $e_{\text{best}} := e$
    if $e_{\text{best}} < e_{\text{allbest}}$ **then**
      $V := V \cup \{v_{\text{best}}\}$
      $e_{\text{allbest}} := e_{\text{best}}$
  return V

Backward Search

1: **procedure** SELECTVARS-BACKWARD($\mathcal{D}^{\text{train}\prime} \subseteq \mathbb{R}^M \times \mathcal{Y}, \text{err}, \mathcal{A}$)
2: $(\mathcal{D}^{\text{train}}, \mathcal{D}^{\text{val}}) := \text{split}(\mathcal{D}^{\text{train}\prime})$
3: $V := \{1, 2, \ldots, M\}$
4: $e_{\text{allbest}} := \text{err}(\mathcal{A}(\pi_V(\mathcal{D}^{\text{train}})), \pi_V(\mathcal{D}^{\text{val}}))$
5: $v_{\text{best}} := 1$
6: **while** $v_{\text{best}} \neq 0$ **do**
7: $v_{\text{best}} := 0$
8: $e_{\text{best}} := e_{\text{allbest}}$
9: **for** $v \in V$ **do**
10: $V' := V \backslash \{v\}$
11: $\hat{y} := \mathcal{A}(\pi_{V'}(\mathcal{D}^{\text{train}}))$
12: $e := \text{err}(\hat{y}, \pi_{V'}(\mathcal{D}^{\text{val}}))$
13: **if** $e < e_{\text{best}}$ **then**
14: $v_{\text{best}} := v$
15: $e_{\text{best}} := e$
16: **if** $e_{\text{best}} < e_{\text{allbest}}$ **then**
17: $V := V \backslash \{v_{\text{best}}\}$
18: $e_{\text{allbest}} := e_{\text{best}}$
19: **return** V

Sequential Search with Variable Importance Heuristics

- Forward and backward search has to learn many models.
  - forward search: 1, 2, 3, ...
  - backward search: M, M-1, M-2, ...
- Further simplication: use a sequential search.
- Use a heuristics to assess **variable importance** once (without context)
  - e.g., the error of the single-variable model:

$$\text{imp}(m) := \text{err}(\mathcal{A}(\pi_{\{m\}}(\mathcal{D}^{\text{train}})), \mathcal{D}^{\text{val}})$$

- Add variables in order of increasing heuristics.
- Usually a full sequential sweep through all variables is done.
  - No difference between Forward and Backward Search.
- Faster, but even less reliable than forward/backward search.

Sequential Search

procedure SELECTVARS-SEQ($\mathcal{D}^{\text{train}\prime} \subseteq \mathbb{R}^M \times \mathcal{Y}$, err, $\mathcal{A}$, imp)
  $(\mathcal{D}^{\text{train}}, \mathcal{D}^{\text{val}}) := \text{split}(\mathcal{D}^{\text{train}\prime})$
  $\mathcal{V} := \text{sort-increasing}(\{1, 2, \ldots, M\}, \text{imp})$
  $V := \emptyset$
  $e_{\text{best}} := \text{err}(\mathcal{A}(\pi_V(\mathcal{D}^{\text{train}})), \pi_V(\mathcal{D}^{\text{val}}))$
  $m_{\text{best}} := 1$
  for $m = 1, \ldots, M$ **do**
    $v := \mathcal{V}_m$
    $V := V \cup \{v\}$
    $\hat{y} := \mathcal{A}(\pi_V(\mathcal{D}^{\text{train}}))$
    $e := \text{err}(\hat{y}, \pi_V(\mathcal{D}^{\text{val}}))$
    if $e < e_{\text{best}}$ **then**
      $m_{\text{best}} := m$
      $e_{\text{best}} := e$
  $V := \{1, 2, \ldots, m_{\text{best}}\}$
  return V

Outline

Minimizing a Function via Coordinate Descent (CD)

Given a function $f : \mathbb{R}^N \to \mathbb{R}$, find β with minimal $f(\beta)$.

- Use the coordinate axes as descent direction
  - first β_1-axis, then β_2-axis, etc. (cyclic)
  - **one-dimensional subproblems**:

$$g_n(\beta) := \arg\min_{\beta_n \in \mathbb{R}} f(\beta_n; \beta_{-n}) := \arg\min_{\beta' \in \mathbb{R}} f(\beta_1, \ldots, \beta_{n-1}, \beta', \beta_{n+1}, \ldots, \beta_N)$$

- Coordinate Descent can be fast if solving the one-dimensional subproblems can be done analytically.
  - For smooth f, one needs to solve

$$\frac{\partial f(\beta_n; \beta_{-n})}{\partial \beta_n} \stackrel{!}{=} 0$$

  - Then also no step length is required !

Note: $\beta_{-n} := (\beta_1, \ldots, \beta_2, \ldots, \beta_{n-1}, \beta_{n+1}, \ldots, \beta_N)$ is the vector without the n-th element for a vector $\beta \in \mathbb{R}^N$.

Coordinate Descent

1: **procedure**
MINIMIZE-CD($f : \mathbb{R}^N \to \mathbb{R}, g, \beta^{(0)} \in \mathbb{R}^N, i_{\max} \in \mathbb{N}, \epsilon \in \mathbb{R}^+$)
2: **for** $i := 1, \ldots, i_{\max}$ **do**
3: $\beta^{(i)} := \beta^{(i-1)}$
4: **for** $n := 1, \ldots, N$ **do**
5: $\beta_n^{(i)} := g_n(\beta_{-n}^{(i)})$
6: **if** $f(\beta^{(i-1)}) - f(\beta^{(i)}) < \epsilon$ **then**
7: **return** $\beta^{(i)}$
8: **error** "not converged in $i_{\max}$ iterations"

g solves g_n for the n-th one-dimensional subproblem

$$g_n(\beta_1, \ldots, \beta_{n-1}, \beta_{n+1}, \ldots, \beta_N) := \arg\min_{\beta' \in \mathbb{R}} f(\beta_1, \ldots, \beta_{n-1}, \beta', \beta_{n+1}, \ldots, \beta_N)$$

Example: Simple Quadratic Function

Minimize

$$f(\beta_1, \beta_2) := \beta_1^2 + \beta_2^2 + \beta_1\beta_2$$

One dimensional problem for β_1:

$$\begin{aligned} f(\beta_1; \beta_2) =& \beta_1^2 + \beta_2^2 + \beta_1\beta_2 \\ \frac{\partial f}{\partial \beta_1}(\beta_1; \beta_2) =& 2\beta_1 + \beta_2 \stackrel{!}{=} 0 \end{aligned}$$

$$\rightsquigarrow \beta_1 = -\frac{1}{2}\beta_2$$

$$\text{i.e., } g_1(\beta_2) := -\frac{1}{2}\beta_2$$

and analogous for β_2:

$$g_2(\beta_1) := -\frac{1}{2}\beta_1$$

Example: Simple Quadratic Function

Minimize

$$f(\beta_1, \beta_2) := \beta_1^2 + \beta_2^2 + \beta_1\beta_2, \quad \beta^{(0)} := (1, 1)$$

$$g_1(\beta_2) := -\frac{1}{2}\beta_2, \quad g_2(\beta_1) := -\frac{1}{2}\beta_1$$

i	$\beta^{(i)}$ before	n	$g_n(\beta^{(i)})$	$\beta^{(i-1)}$ after
1	$(1, 1)$	1	$-1/2$	$(-1/2, 1)$
	$(-1/2, 1)$	2	$1/4$	$(-1/2, 1/4)$
2	$(-1/2, 1/4)$	1	$-1/8$	$(-1/8, 1/4)$
	$(-1/8, 1/4)$	2	$1/16$	$(-1/8, 1/16)$
$\vdots$				

Note: Minimize $f(\beta_1, \beta_2) := \beta_1^2 + \beta_2^2$ via CD yourself. What is different? Why?

Learn Linear Regression via CD

Minimize

$$f(\hat{\beta}) := ||y - X\hat{\beta}||^2 \propto \hat{\beta}^T X^T X \hat{\beta} - 2y^T X \hat{\beta}$$

$$\begin{aligned} f(\hat{\beta}_m; \hat{\beta}_{-m}) &= x_m^T x_m \hat{\beta}_m^2 + 2\hat{\beta}_{-m}^T X_{-m}^T x_m \hat{\beta}_m + \hat{\beta}_{-m}^T X_{-m}^T X_{-m} \hat{\beta}_{-m} \\ &\quad - 2y^T x_m \hat{\beta}_m - 2y^T X_{-m} \hat{\beta}_{-m} \\ &\propto x_m^T x_m \hat{\beta}_m^2 - 2(y - X_{-m}\hat{\beta}_{-m})^T x_m \hat{\beta}_m \end{aligned}$$

$$\frac{\partial f(\hat{\beta}_m; \hat{\beta}_{-m})}{\partial \hat{\beta}_m} \stackrel{!}{=} 0 \rightsquigarrow \hat{\beta}_m = \frac{(y - X_{-m}\hat{\beta}_{-m})^T x_m}{x_m^T x_m}$$

Note: $x_m := X_{\cdot,m}$ denotes the m-th column of X,
X_{-m} denotes the matrix X without column m.

Learn Linear Regression via CD

1: **procedure** LEARN-LINREG-CD($\mathcal{D}^{\text{train}} := \{(x_1, y_1), \ldots, (x_N, y_N)\}, i_{\max} \in \mathbb{N}, \epsilon \in \mathbb{R}^+$)

2: $X := (x_1, x_2, \ldots, x_N)^T$

3: $y := (y_1, y_2, \ldots, y_N)^T$

4: $\hat{\beta}_0 := (0, \ldots, 0)$

5: $\hat{\beta} :=$ MINIMIZE-CD($f(\hat{\beta}) := (y - X\hat{\beta})^T(y - X\hat{\beta})$, $g(\hat{\beta}_m; \hat{\beta}_{-m}) := \frac{(y - X_{-m}\hat{\beta}_{-m})^T x_m}{x_m^T x_m}$, $\hat{\beta}_0, i_{\max}, \epsilon$)

6: **return** $\hat{\beta}$

Note: $x_m := X_{\cdot,m}$ denotes the m-th column of X, X_{-m} denotes the matrix X without column m.

Outline

L1 Regularization

Let X be the predictor matrix and y the target vector,
$\hat{\beta}$ the model parameters,
$\hat{y}$ the model predictions and
ℓ the loss/error.

L2 regularization:

$$f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X)) + \lambda ||\hat{\beta}||_2^2 = \ldots + \lambda \sum_{p=1}^{P} \beta_p^2$$

L1 regularization:

$$f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X)) + \lambda ||\hat{\beta}||_1 = \ldots + \lambda \sum_{p=1}^{P} |\hat{\beta}_p|$$

Why L1 Regularization?

$$\text{min.} \quad f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X)) + \lambda ||\hat{\beta}||_1$$
$$\hat{\beta} \in \mathbb{R}^P$$

is equivalent to

$$\text{min.} \quad f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X))$$
$$||\hat{\beta}||_1 \leq B$$
$$\hat{\beta} \in \mathbb{R}^P$$

with

$$B := ||\hat{\beta}^*||_1$$

Note: $\hat{\beta}^*$ denotes the optimal parameters. Thus this equivalence provides insight, but cannot (yet) be used to solve the problem.

Why L1 Regularization?

$$\min_{\hat{\beta} \in \mathbb{R}^P} f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X)) + \lambda ||\hat{\beta}||_1$$

is equivalent to

$$\min_{\hat{\beta} \in \mathbb{R}^P} f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X))$$
$$||\hat{\beta}||_1 \leq B$$

with

$$B := ||\hat{\beta}^*||_1$$

$$\min_{\hat{\beta} \in \mathbb{R}^P} f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X)) + \lambda ||\hat{\beta}||_2^2$$

is equivalent to

$$\min_{\hat{\beta} \in \mathbb{R}^P} f(\hat{\beta}) := \ell(y, \hat{y}(\hat{\beta}, X))$$
$$||\hat{\beta}||_2^2 \leq B$$

with

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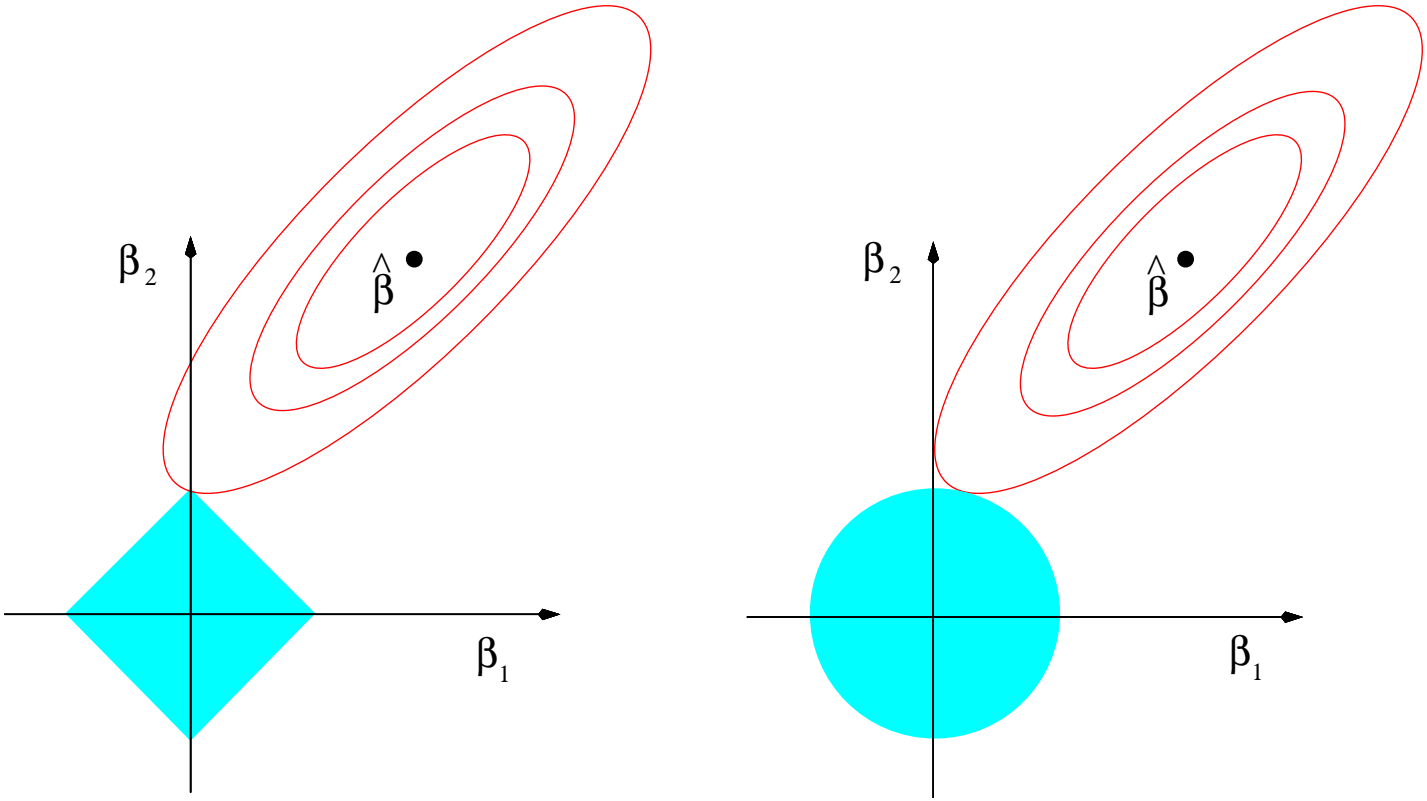
Why L1 Regularization?



source: [HTFF05, p. 90]

Regularized Linear Regression

Let X the predictor matrix and y the target vector,
$\hat{\beta}$ the linear regression model parameters,
$\hat{y} := X\hat{\beta}$ the linear regression model predictions and
$\ell(y, \hat{y}) := ||y - \hat{y}||_2^2$ the RSS loss/error.

L2 Regularized Linear Regression (**Ridge Regression**):

$$\begin{aligned} f(\hat{\beta}) &:= \ell(y, \hat{y}(\hat{\beta}, X)) + \lambda||\hat{\beta}||_2^2 \\ &\propto \hat{\beta}^T X^T X \hat{\beta} - 2y^T X \hat{\beta} + \lambda \hat{\beta}^T \hat{\beta} \\ &= \hat{\beta}^T (X^T + \lambda^{\frac{1}{2}} I)(X + \lambda^{\frac{1}{2}} I)\hat{\beta} - 2y^T X \hat{\beta} \end{aligned}$$

- L2 regularized problem has same structure as unregularized one.
- All learning algorithms work seamlessly.

Regularized Linear Regression

Let X the predictor matrix and y the target vector,
$\hat{\beta}$ the linear regression model parameters,
$\hat{y} := X\hat{\beta}$ the linear regression model predictions and
$\ell(y, \hat{y}) := ||y - \hat{y}||_2^2$ the RSS loss/error.

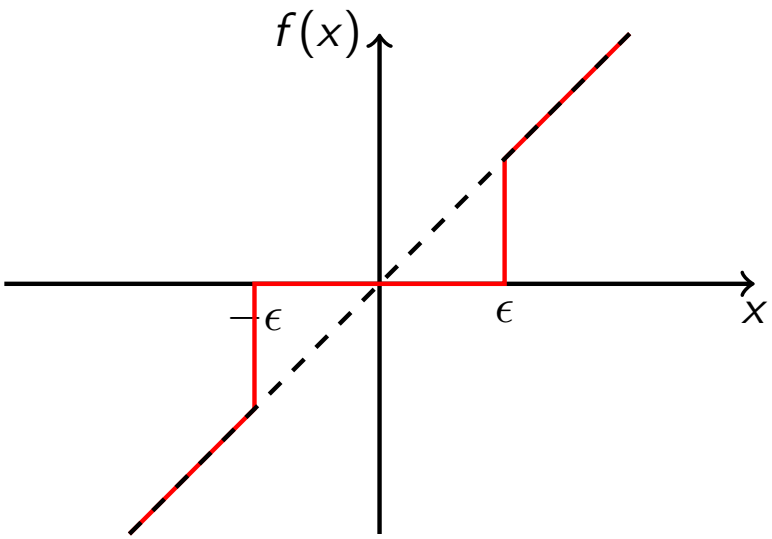
L1 regularized Linear Regression (**Lasso**):

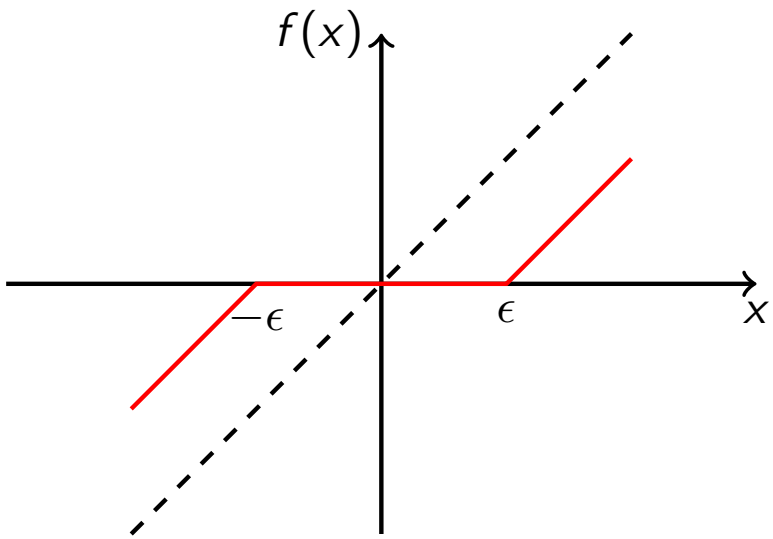
$$f(\beta) := \ell(y, \hat{y}) + \lambda||\beta||_1$$

$$\propto \hat{\beta}^T X^T X\hat{\beta} - 2y^T X\hat{\beta} + \lambda \sum_{m=1}^{M} |\beta_m|$$

- L1 regularized problem has new terms $|\beta_m|$.
  - Esp. non-differentiable at 0.
- All learning algorithms seen so far do not work.
  - Solving SLE is not applicable.
  - Gradient Descent does not work.

Hard & Soft Thresholding





$$\text{hard}(x,\epsilon) := \begin{cases} x, & \text{if } |x| > \epsilon \\ 0, & \text{else} \end{cases}$$

$$\text{soft}(x,\epsilon) := \begin{cases} x-\epsilon, & \text{if } x > \epsilon \\ 0, & \text{if } |x| \leq \epsilon \\ x+\epsilon, & \text{if } x < -\epsilon \end{cases}$$

Coordinate Gradient for L1 Regularized Linear Regression

$$f(\hat{\beta}) := \hat{\beta}^T X^T X \hat{\beta} - 2y^T X \hat{\beta} + \lambda \sum_{m=1}^{M} |\beta_m|$$

$$f(\hat{\beta}_m; \hat{\beta}_{-m}) \propto x_m^T x_m \hat{\beta}_m^2 - 2(y - X_{-m}\hat{\beta}_{-m})^T x_m \hat{\beta}_m + \lambda|\beta_m|$$

$$\frac{\partial f(\hat{\beta}_m; \hat{\beta}_{-m})}{\partial \hat{\beta}_m} \stackrel{!}{=} 0 \rightsquigarrow \hat{\beta}_m = \frac{(y - X_{-m}\hat{\beta}_{-m})^T x_m - \frac{1}{2}\lambda}{x_m^T x_m}, \quad \hat{\beta}_m > 0$$

$$\hat{\beta}_m = \frac{(y - X_{-m}\hat{\beta}_{-m})^T x_m + \frac{1}{2}\lambda}{x_m^T x_m}, \quad \hat{\beta}_m < 0$$

$$\rightsquigarrow \hat{\beta}_m = \text{soft}\left(\frac{(y - X_{-m}\hat{\beta}_{-m})^T x_m}{x_m^T x_m}, \frac{\frac{1}{2}\lambda}{x_m^T x_m}\right)$$

Note: LASSO = Least Absolute Selection and Shrinkage Operator.

Learn L1-regularized Linear Regression via CD (Shooting Algorithm)

1: **procedure** LEARN-LINREG-L1REG-CD($\mathcal{D}^{\text{train}} := \{(x_1, y_1), \ldots, (x_N, y_N)\}, \lambda \in \mathbb{R}^+, i_{\max} \in \mathbb{N}, \epsilon \in \mathbb{R}^+$)

2: $X := (x_1, x_2, \ldots, x_N)^T$

3: $y := (y_1, y_2, \ldots, y_N)^T$

4: $\hat{\beta}_0 := (0, \ldots, 0)$

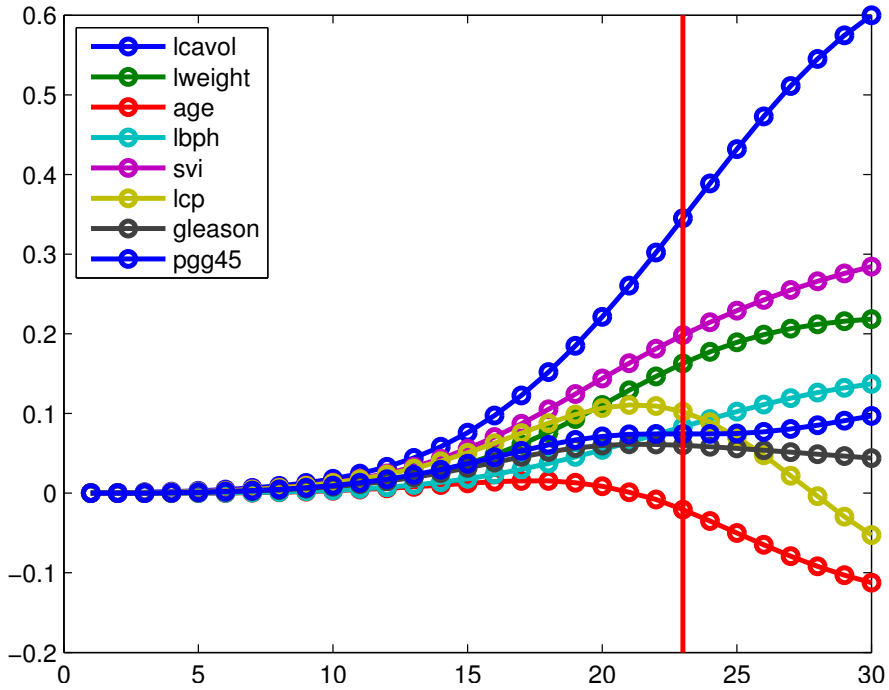
5: $\hat{\beta} :=$ MINIMIZE-CD($f(\hat{\beta}) := (y - X\hat{\beta})^T(y - X\hat{\beta}) + \lambda||\beta||_1$, $g(\hat{\beta}_m; \hat{\beta}_{-m}) := \text{soft}(\frac{(y - X_{-m}\hat{\beta}_{-m})^T x_m}{x_m^T x_m}, \frac{\frac{1}{2}\lambda}{x_m^T x_m})$, $\hat{\beta}_0, \alpha, i_{\max}, \epsilon$)

6: **return** $\hat{\beta}$

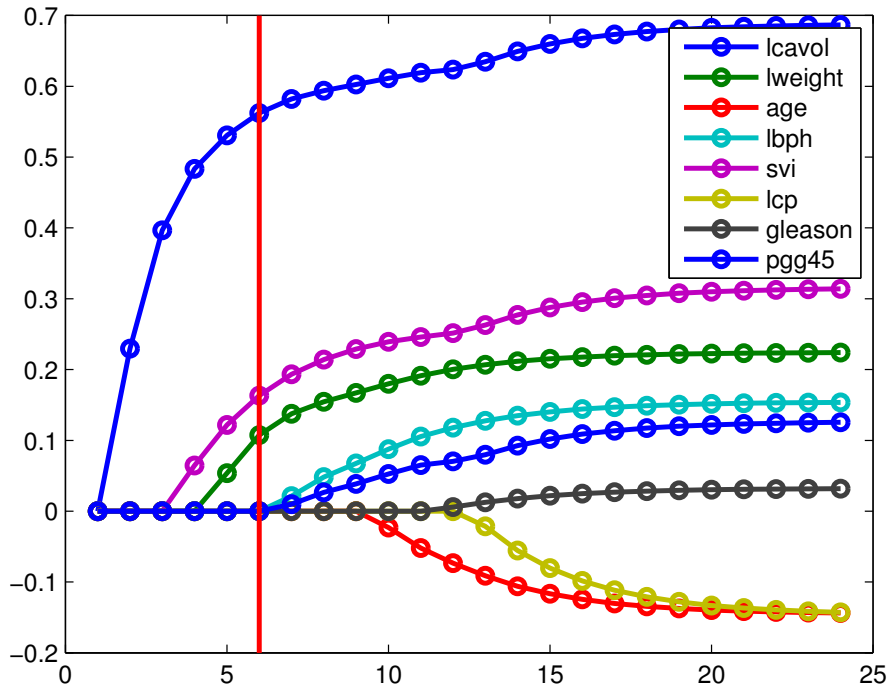
Note: $x_m := X_{\cdot,m}$ denotes the m-th column of X, X_{-m} denotes the matrix X without column m.

Regularization Paths

L2 regularization



L1 regularization



x-axis: bound B on parameter size.
y-axis: parameter $\hat{\theta}$.

source: [Mur12, p. 437]

Summary

- **High-dimensional data** poses problems as many parameters have to be estimated from comparable few instances.
- Non-linear effects can be captured by **derived predictor variables**.
  - e.g., in **polynomial models**.
  - making even originally low-dimensional data high-dimensional.
- Relevant variables can be searched explicitly through a greedy **forward search** and **backward search**.
- To minimize a function, **coordinate descent** cyclicly chooses the coordinate axes as descent direction.
  - efficient, if the **one-dimensional subproblems** can be solved analytically.
  - does need no step length.
- Variable selection also can be accomplished by **L1 regularization**.
  - **L1 regularized linear regression (LASSO)** can be learned by coordinate descent (**shooting algorithm**).

Further Readings

- [JWHT13, chapter 6], [Mur12, chapter 13], [HTFF05, chapter 3.3–8].

References

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