

# Machine Learning

## B. Unsupervised Learning

### B.3 Frequent Pattern Mining

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# Outline

1. The Frequent Itemset Problem
2. Breadth First Search: Apriori Algorithm
3. Depth First Search: Eclat Algorithm
4. Supervised Pattern Mining

# Syllabus

Tue. 21.10. (1) 0. Introduction

## **A. Supervised Learning**

Wed. 22.10. (2) A.1 Linear Regression

Tue. 28.10. (3) A.2 Linear Classification

Wed. 29.10. (4) A.3 Regularization

Tue. 4.11. (5) A.4 High-dimensional Data

Wed. 5.11. (6) A.5 Nearest-Neighbor Models

Tue. 11.11. (7) A.6 Decision Trees

Wed. 12.12. (8) A.7 Support Vector Machines

Tue. 18.11. (9) A.8 A First Look at Bayesian and Markov Networks

## **B. Unsupervised Learning**

Wed. 19.11. (10) B.1 Clustering

Tue. 25.11. (11) B.2 Dimensionality Reduction

Wed. 26.11. (12) B.3 Frequent Pattern Mining

## **C. Reinforcement Learning**

Tue. 2.12. (13) C.1 State Space Models

Wed. 3.12. (14) C.2 Markov Decision Processes

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1. The Frequent Itemset Problem
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# Market Basket Analysis

cid	beer	bread	icecream	milk	pampers	pizza
1	+	-	-	+	+	+
2	+	+	-	-	+	+
3	+	-	+	-	+	+
4	-	+	-	+	-	+
5	-	+	+	+	-	-
6	+	+	-	+	+	-

# Market Basket Analysis

Association rules in large transaction datasets:

- ▶ look for products frequently bought together (**frequent itemsets**).

Examples:

- ▶ {beer, pampers, pizza} (support=0.5)
- ▶ {bread, milk} (support=0.5)

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# Market Basket Analysis

Association rules in large transaction datasets:

- ▶ look for products frequently bought together (**frequent itemsets**).
- ▶ look for rules in buying behavior (**association rules**)

Examples:

- ▶ {beer, pampers, pizza} (support=0.5)
- ▶ {bread, milk} (support=0.5)
- ▶ If beer and pampers, then pizza (confidence= 0.75)
- ▶ If bread, then milk (confidence=0.75)

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2	+	+	-	-	+	+
3	+	-	+	-	+	+
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5	-	+	+	+	-	-
6	+	+	-	+	+	-

# Transaction Data, Frequency vs Support

Let  $I$  be a set called **set of items**.

A subset  $X \subseteq I$  is called **itemset**.

Let  $\mathcal{D} \subseteq \mathcal{P}(I)$  be a set of subsets of  $I$  called **transaction data set**.

An element  $X \in \mathcal{D}$  is called **transaction**.

The **frequency of a subset  $X$  in a data set  $\mathcal{D}$**  is (as always)

$$\text{freq}(X; \mathcal{D}) := |\{Y \in \mathcal{D} \mid X = Y\}|$$

Note:  $\mathcal{D}$  really is a multiset: a transaction could occur multiple times in  $\mathcal{D}$  and then is counted as often as it occurs in computing frequency and support.



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The **support of a subset  $X$  in a data set  $\mathcal{D}$**  is the number of transactions it is a subset of:

$$\text{sup}(X; \mathcal{D}) := |\{Y \in \mathcal{D} \mid X \subseteq Y\}|$$

Note:  $\mathcal{D}$  really is a multiset: a transaction could occur multiple times in  $\mathcal{D}$  and then is counted as often as it occurs in computing frequency and support.

# Transaction Data, Frequency vs Support / Example

$$I := \{1, 2, 3, 4, 5, 6, 7\}$$

$$\mathcal{D} := \left\{ \begin{array}{l} \{ 1, 3, 5 \} \\ \{ 1, 2, 3, 5 \} \\ \{ 1, 3, 4, 6 \} \\ \{ 1, 3, 4, 5, 7 \} \\ \{ 2, 4, 7 \} \\ \{ 1, 3, 5 \} \\ \{ 1, 5, 7 \} \\ \{ 1, 2, 3, 4, 5 \} \end{array} \right\}$$

$$\text{freq}(\{1, 3, 5\}) = 2$$

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$$\text{freq}(\{1, 3, 5\}) = 2$$

$$\text{sup}(\{1, 3, 5\}) = 5$$

# The Frequent Itemsets Problem

Given

- ▶ a set  $I$  (called **set of items**),
- ▶ a set  $\mathcal{D} \subseteq \mathcal{P}(I)$  of subsets of  $I$  called **transaction data set**, and
- ▶ a number  $s \in \mathbb{N}$  called **minimum support**,

find all subsets  $X$  of  $I$  whose support exceeds the given minimum support

$$\text{sup}(X; \mathcal{D}) := |\{Y \in \mathcal{D} \mid X \subseteq Y\}| \geq s$$

and their support.

Such subsets  $X \subseteq I$  with  $\text{sup}(X) \geq s$  are called **frequent** (w.r.t. minimum support  $s$  in data set  $\mathcal{D}$ ).

# Subsets of Frequent Itemsets are Frequent

Obviously, the support of a subset is at least as large as the one of any superset:

$$\text{for all } X \subseteq Y \subseteq I : \quad \text{sup } X \geq \text{sup } Y$$

For a frequent set, all its subsets are frequent.

# The **Maximal** Frequent Itemsets Problem

Given

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- ▶ a set  $\mathcal{D} \subseteq \mathcal{P}(I)$  of subsets of  $I$  called **transaction data set**, and
- ▶ a number  $s \in \mathbb{N}$  called **minimum support**,

find all **maximal** subsets  $X$  of  $I$  whose support exceeds the given minimum support

$$\text{sup}(X; \mathcal{D}) := |\{Y \in \mathcal{D} \mid X \subseteq Y\}| \geq s$$

and their support.

i.e., there exists no frequent superset of  $X$ , i.e., no set  $X' \subseteq I$  with

- ▶  $\text{sup}(X'; \mathcal{D}) \geq s$  and
- ▶  $X \subsetneq X'$

# Surprising Frequent Itemsets

Example:

Assume item 1 occurs in 50% of all transactions and  
item 2 occurs in 25% of all transactions.

- ▶ Is it surprising that itemset  $\{1, 2\}$  occurs in 12.5% of all transactions?
- ▶ Does a relative support of 12.5% of itemset  $\{1, 2\}$  signal a strong association between both items?

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$$p(\{1\} \subseteq X) = 0.5, \quad p(\{2\} \subseteq X) = 0.25$$

If both items occur independently

$$\rightsquigarrow p(\{1, 2\} \subseteq X) = p(\{1\} \subseteq X)p(\{2\} \subseteq X) = 0.125$$



# Surprising Frequent Itemsets: Lift

$$\text{lift}(X) := \frac{\frac{1}{N} \sup X}{\prod_{x \in X} \frac{1}{N} \sup \{x\}}, \quad N := |\mathcal{D}|$$

- ▶  $\text{lift}(X) > 1$ : itemset  $X$  is more frequent than expected (positive association)
- ▶  $\text{lift}(X) < 1$ : itemset  $X$  is less frequent than expected (negative association)

Example:

$$\text{lift}(\{1, 2\}) = \frac{\frac{1}{N} \sup \{1, 2\}}{\frac{1}{N} \sup \{1\} \frac{1}{N} \sup \{2\}} = \frac{0.125}{0.5 \cdot 0.25} = 1$$

# Association Rules

Sometimes one is interested to extract if-then rules of the type

if a transaction contains items  $X$ , then it also contains items  $Y$   
all transactions containing  $X$  also contain  $Y$

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if a transaction contains items  $X$ , then it usually also contains items  $Y$   
most transactions containing  $X$  also contain  $Y$

Find all **association rules**  $(X, Y)$ ,  $X, Y \subseteq I, X \cap Y = \emptyset$  that

- ▶ are **exact enough** / hold in most cases:  
high **confidence**, confidence exceeds minimum confidence  $c$ :

$$\text{conf}(X, Y) := \frac{\text{sup}(X \cup Y)}{\text{sup}(X)} \geq c$$

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 high **confidence**, confidence exceeds minimum confidence  $c$ :

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- ▶ are **general enough** / frequently applicable:  
 high **support**, support exceeds minimum support  $s$ :

$$\text{sup}(X, Y) := \text{sup}(X \cup Y) \geq s$$

# Finding All Association Rules

To find all association rules that

- ▶ exceed a given minimum confidence  $c$  and
- ▶ exceed a given minimum support  $s$

it is sufficient

1. to find all frequent itemsets that exceed a given minimum support  $s$  **and their supports** and then

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2. to split each frequent itemset  $Z$  in any two subsets  $X, Y$  s.t. the rule  $(X, Y)$  meets the minimum confidence requirement.

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2. to split each frequent itemset  $Z$  in any two subsets  $X, Y$  s.t. the rule  $(X, Y)$  meets the minimum confidence requirement.
  - ▶ start with rule  $(Z, \emptyset)$  with confidence 1,
  - ▶ iteratively move one element from body to head and retain only those rules that meet the minimum confidence requirement.

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# Nominal Data as Transaction Data

Data consisting of only nominal variables can be naturally represented as transaction data.

Example:

- ▶  $X_1$  :  $\text{dom}(X_1) = \{\text{red, green, blue}\}$ : border color,
- ▶  $X_2$  :  $\text{dom}(X_2) = \{\text{red, green, blue}\}$ : area color,
- ▶  $X_3$  :  $\text{dom}(X_3) = \{\text{triangle, rectangle, circle}\}$ : shape,
- ▶  $X_4$  :  $\text{dom}(X_4) = \{\text{small, medium, large}\}$ : size.

Vector representation:

$$x = (\text{green, blue, rectangle, large})$$



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Itemset representation:

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# Numerical / Any Data as Transaction Data

To represent data with numerical variables as transaction data, numerical variables have to be **discretized** to ordinal/nominal levels.

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Discretization:

- ▶  $X'_4$  :  $\text{dom}(X_4) = \{1, 2, 3\}$ : diameter.
  - ▶  $X'_4 = 1$  :  $\Leftrightarrow X_4 < 10$ ,
  - ▶  $X'_4 = 2$  :  $\Leftrightarrow 10 \leq X_4 < 20$ ,
  - ▶  $X'_4 = 3$  :  $\Leftrightarrow 20 \leq X_4$ .

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Itemset representation:  $x = \{\text{border.green, area.blue, rectangle, diameter.2}\}$

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# Discretization Schemes

- ▶ **equi-range**:
  - ▶ split the domain of the variable in  $k$  intervals of same size
- ▶ **equi-volume** (w.r.t. a sample/dataset  $\mathcal{D}$ ):
  - ▶ split the domain of the variable in  $k$  intervals with same frequency

Discretization of numerical variables can be useful in many other contexts.

- ▶ e.g., discretization can be used to model non-linear dependencies.

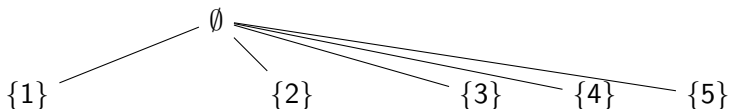
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# Naive Breadth First Search

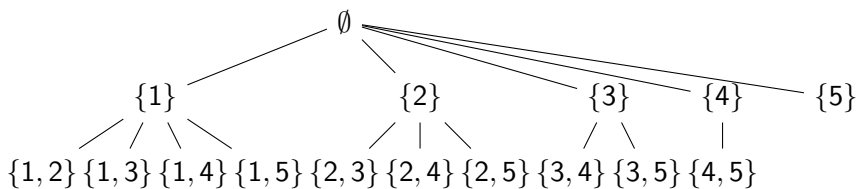
$\emptyset$

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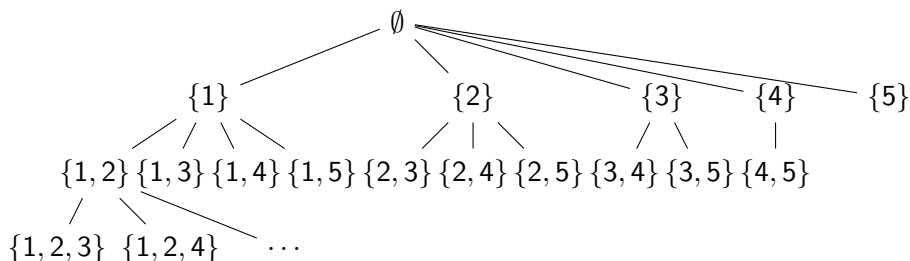




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# Naive Breadth First Search

To find all frequent itemsets, one can employ **Breadth First Search**:

1. start with all **frequent itemsets**  $F_0$  of size  $k := 0$ :

$$F_0 := \{\emptyset\}$$

2. for each  $k = 1, 2, \dots, |I|$ : find all frequent itemsets  $F_k$  of size  $k$ :
  - 2.1 extend frequent itemsets  $F_{k-1}$  to **candidates**  $C_k$ :

$$C_k := \{X \cup \{y\} \mid X \in F_{k-1}, y \in I\}$$

- 2.2 **count the support** of all candidates

$$s_X := \text{sup}(X, \mathcal{D}), \quad X \in C_k$$

- 2.3 retain only frequent candidates as **frequent itemsets**  $F_k$ :

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# Improvement 1: Fewer Candidates

- ▶  $k$ -candidates can be created from different  $k - 1$ -subsets:

$$\{1, 3, 4, 7\} = \{1, 3, 4\} \cup \{7\} = \{1, 3, 7\} \cup \{4\}$$

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↪ **fuse candidates** from two frequent itemsets from  $F_{k-1}$ ,  
check all other subsets of size  $k - 1$ .

## Ordered Itemsets, Prefix and Head

Let us fix an order on the items  $I$  (e.g.,  $<$  for  $I \subseteq \mathbb{N}$ ).

Let  $X \subseteq I$  be an itemset, then

$$h(X) := \max X$$

is called **the head of  $X$**  and

$$p(X) := X \setminus \{h(X)\}$$

is called **the prefix of  $X$** .

Example:

$$h(\{1, 3, 4, 7\}) = 7$$

$$p(\{1, 3, 4, 7\}) = \{1, 3, 4\}$$

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For two  $k - 1$ -itemsets  $X, Y$ :

$X \cup Y$  yields a  $k$ -candidate  
that extends  $X$  by a larger item

$$\left. \vphantom{\begin{array}{l} X \cup Y \text{ yields a } k\text{-candidate} \\ \text{that extends } X \text{ by a larger item} \end{array}} \right\} \iff p(X) = p(Y) \text{ and } h(X) < h(Y)$$

# Improved Breadth First Search (1/2)

To find all frequent itemsets:

1. start with all **frequent itemsets**  $F_0$  of size  $k := 0$ :

$$F_0 := \{\emptyset\}$$

2. for  $k = 1, 2, \dots, |I|$ , while  $F_{k-1} \neq \emptyset$ :

- 2.1 extend frequent itemsets  $F_{k-1}$  to **candidates**  $C_k$ :

$$C'_k := \{X \cup \{h(Y)\} \mid X, Y \in F_{k-1}, p(X) = p(Y), h(X) < h(Y)\}$$

- 2.2 **retain only candidates with frequent  $k - 1$ -subsets (pruning):**

$$C_k := \{X \in C'_k \mid \forall x \in X : X \setminus \{x\} \in F_{k-1}\}$$

- 2.3 **count the support** of all candidates

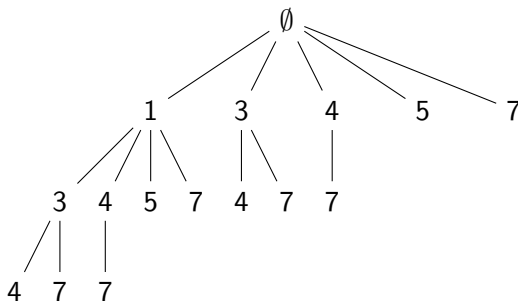
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- 2.4 retain only frequent candidates as **frequent itemsets**  $F_k$ :

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## Improvement 2: Compact Representation and Fast Candidate Creation

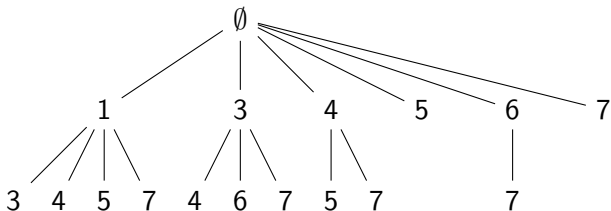
- ▶ all frequent itemsets found so far and the latest candidates can be represented compactly in a **trie**:



- ▶ every node is labeled with a single item,
- ▶ every node represents the subset containing all items along the path to the root.

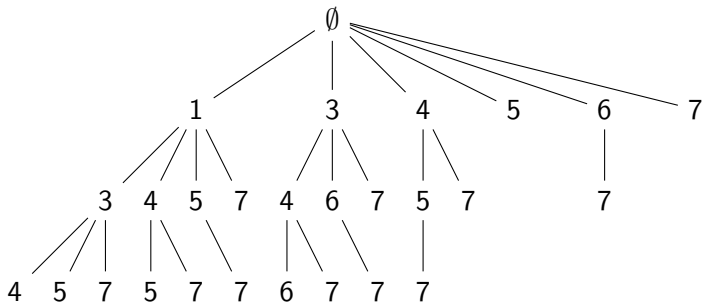
## Improvement 2: Compact Representation and Fast Candidate Creation

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## Improvement 3: Fewer Subset Checks for Counting

- ▶ computing the support of all candidates  $C_k$  naively requires  $|C_k|$  passes over the database  $\mathcal{D}$ .
- ▶ instead, count each transaction  $X$  into the candidate trie:
  - ▶ start at the root  $N$ :  $\text{count}(X, \text{root})$ .
  - ▶  $\text{count}(X, N)$ : count transaction  $X$  into trie rooted at  $N$ :
    1. if  $N$  is a leaf node at depth  $k$ :

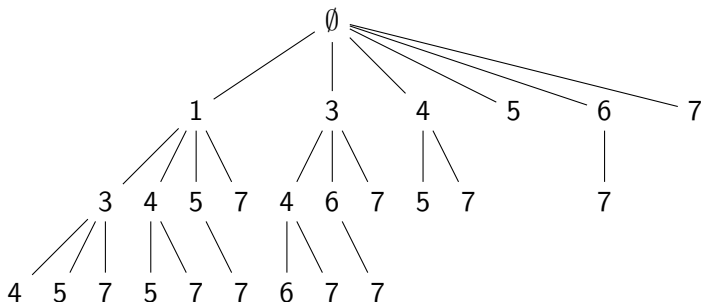
$$s_N := s_N + 1;$$

2. else for all child nodes  $M$  of  $N$  with  $\text{item}(M) \in X$ :

$$\text{count}(X, M)$$

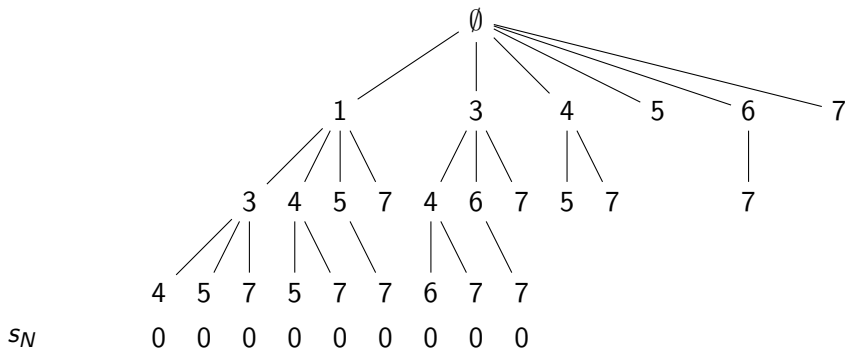
# Example: Counting Transaction into Candidate Trie

Count {1, 3, 5, 7, 8} into the trie:



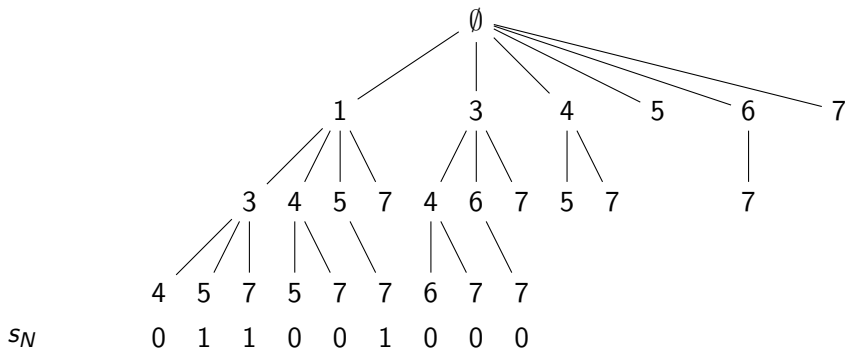
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## Improved Breadth First Search (2/2): Apriori

To find all frequent itemsets with minimum support  $s$  in database  $\mathcal{D}$ :

1. create a trie  $T$  with just the root node  $R$  without label.
2. for  $x \in I$ :
  - add a node  $N$  to  $T$  with label  $x$  and parent  $R$ .
3. for  $k := 1, 2, \dots, |I|$ , while  $T$  has nodes at depth  $k$ :
  - 3.1 for  $X \in \mathcal{D}$ :
    - count( $X, R$ ). [computing  $N.s$  for nodes at depth  $k$ ]
  - 3.2 for all nodes  $N$  of  $T$  at depth  $k$ :
    - if  $N.s < s$ , remove node  $N$ .
  - 3.3 for all nodes  $N$  of  $T$  at depth  $k$ :
    - 3.3.1 for all right-side siblings  $M$  of  $N$ :
      - for all nodes  $L$  on the path from  $N$  to  $R$ :
        - check if the node representing itemset( $N$ )  $\setminus$  {label( $L$ )}  $\cup$  {label( $M$ )} exists
        - if so, add a node  $K$  to  $T$  with the label of  $M$  and parent  $N$ .
4. return  $T$

# Apriori: Sparse Child Arrays

To find all frequent itemsets with minimum support  $s$  in database  $\mathcal{D}$ :

1. create a trie  $T$  with just the unlabeled root node  $R$ .
2. for  $x \in I$ :
  - add a node  $N$  to  $T$  with label  $x$  and parent  $R$ :  $R.\text{child}[x] := N$ .
3. for  $k := 1, 2, \dots, |I|$ , while  $T$  has nodes at depth  $k$ :
  - 3.1 for  $X \in \mathcal{D}$ :
    - count( $X, R$ ). [computing  $N.s$  for nodes at depth  $k$ ]
  - 3.2 for all nodes  $N$  of  $T$  at depth  $k$ :
    - if  $N.s < s$ , remove node  $N$ .
  - 3.3 for all nodes  $N$  of  $T$  at depth  $k$ :
    - 3.3.1 for all right-side siblings  $M$  of  $N$ :
      - for all nodes  $L$  on the path from  $N$  to  $R$ :
        - check if the node representing itemset  $(N) \setminus \{L.\text{label}\} \cup \{M.\text{label}\}$  exists
        - if so, add a node  $K$  to  $T$  with label of  $M$  and parent  $N$ :  
 $N.\text{child}[M.\text{label}] := K$ .
4. return  $T$

# Apriori: Algorithmic Improvements

Scalable Apriori implementations usually employ some further simple tricks:

- ▶ initially, sort items by decreasing frequency
  - ▶ count all item frequencies
  - ▶ recode items s.t. code 0 is the most frequent, code 1 the next most frequent etc.
  - ▶ remove all infrequent items from the database  $\mathcal{D}$ .
  - ▶ this automatically yields  $F_1$  and their supports.
- ▶ count  $C_2$  in a triangular matrix, start trie from level 3 onwards.
- ▶ remove transactions from the database once they contain no frequent itemset of  $F_k$  anymore.
- ▶ branches in the candidate trie without leaf nodes are not used for counting and candidate generation.

# Outline

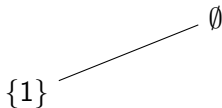
1. The Frequent Itemset Problem
2. Breadth First Search: Apriori Algorithm
3. Depth First Search: Eclat Algorithm
4. Supervised Pattern Mining



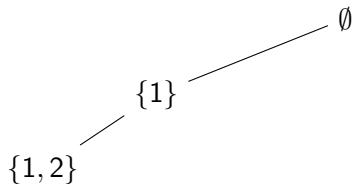
# Naive Depth First Search

$\emptyset$

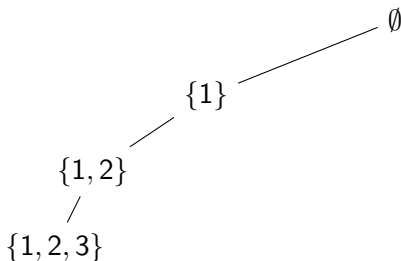
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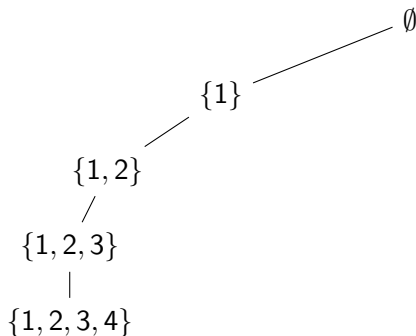
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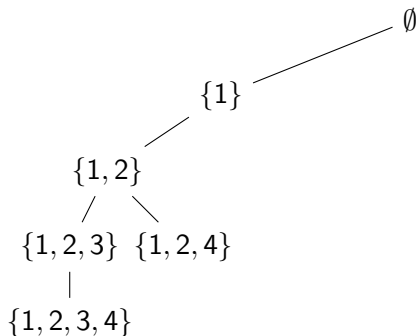
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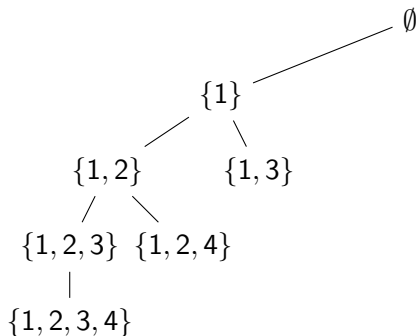
# Naive Depth First Search



# Naive Depth First Search



# Naive Depth First Search



# Naive Depth First Search

To find all frequent itemsets, one can employ **Depth First Search**:

- ▶ start with the empty itemset:

$$F := \{\emptyset\}$$

extend-itemset( $\emptyset$ )

- ▶ extend-itemset( $P$ ):

for all  $y \in I$ :

1. extend current prefix  $P$  to candidate  $X$ :

$$X := P \cup \{y\}$$

2. count the support of candidate  $X$ :

$$s_X := \text{sup}(X, \mathcal{D})$$

3. retain and recursively extend if candidate is frequent:

if  $s_X \geq s$ :

$$F := F \cup \{X\}$$

extend-itemset( $X$ )



## Improvement 1: Fewer Candidates

- ▶  $k$ -candidates can be created from different  $k - 1$ -prefices:

$$\{1, 3, 4, 7\} = \{1, 3, 4\} \cup \{7\} = \{1, 3, 7\} \cup \{4\}$$

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↪ **fuse candidates** from two frequent  $k - 1$ -itemsets,  
check all other subsets of size  $k - 1$ .

# Checking $k - 1$ -subsets in DFS

Checking all  $k - 1$ -subsets:

- ▶ In BFS:
  - ▶ all frequent  $k - 1$ -itemsets are available from last level
  - ▶ no problem
  
- ▶ In DFS:
  - ▶ not all  $k - 1$ -itemsets have been checked yet !
  - ▶ traverse extension items **in decreasing item order**:
    - ▶ ensures that all  $k - 1$ -subsets

$$(i_1, i_2, \dots, i_{l-1}, \hat{i}_l, i_{l+1}, \dots, i_k)$$

are checked before  $(i_1, i_2, \dots, i_{l-1}, i_l, \dots, i_{k-1})$ .

# Improved Depth First Search (1/2)

- ▶ start with the empty itemset:

$$F := \{\emptyset\}, J_{\emptyset} := \{x \in I \mid \text{sup}\{x\} \geq s\}$$

$$\text{extend-itemset}(\emptyset, J_{\emptyset})$$

- ▶  $\text{extend-itemset}(P, J)$ :

for all  $y \in J$  in **decreasing order**:

1. extend current prefix  $P$  to candidate  $X$ :  $X := P \cup \{y\}$
2. **ensure that all  $k - 1$ -subsets are frequent**:

**if  $\exists \ell = 1, \dots, k - 2 : P \setminus \{P_{\ell}\} \cup \{y\} \notin F$ , then skip and go to next  $y$**

3. count the support of candidate  $X$ :  $s_X := \text{sup}(X, \mathcal{D})$
4. retain and recursively extend if candidate is frequent:

if  $s_X \geq s$ :

$$F := F \cup \{X\}$$

$$J_X := \{z \in J \mid z > y, s_{P \cup \{z\}} \geq s\}$$

$\text{extend-itemset}(X, J_X)$



## Improvement 2: Project Data for Fast Support Counting

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- ▶ first idea:
  - ▶ do not check transactions again that do not contain the prefix  $P$
  - ▶  $\rightsquigarrow$  keep a list of transaction IDs that contain the prefix:

$\mathcal{D} = \{X_1, \dots, X_N\}$  full data set

$T(P) := \{t \in \{1, \dots, N\} \mid P \subseteq X_t\}$  transaction cover of  $P$

- ▶ to compute frequency of  $P \cup \{y\}$ ,  
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check only  $P \cup \{y\} \stackrel{?}{\in} X_t$  with  $t \in T(P)$
- ▶ final idea:
  - ▶ compute  $T$  recursively:

$$T(P \cup \{z\} \cup \{y\}) = T(P \cup \{z\}) \cap T(P \cup \{y\})$$

- ▶ store extension items  $z$  together with  $T(P \cup \{z\})$ .

## Improved Depth First Search (2/2): Eclat

- ▶ start with the empty itemset:

$$F := \{\emptyset\}, J_{\emptyset} := \{(x, T(x)) \mid x \in I, |T(x)| \geq s\}$$

$$\text{extend-itemset}(\emptyset, \{1, \dots, N\}, J_{\emptyset})$$

- ▶  $\text{extend-itemset}(P, T_P, J)$ :

for all  $(y, T_y) \in J$  in **decreasing order of**  $y$ :

1. extend current prefix  $P$  to candidate  $X$ :  $X := P \cup \{y\}$
2. **ensure that all  $k - 1$ -subsets are frequent:**

**if  $\exists \ell = 1, \dots, k - 2 : P \setminus \{P_{\ell}\} \cup \{y\} \notin F$ , then skip and go to next  $y$**

3. **compute transaction cover of candidate  $X$ :  $T_X := T_P \cap T_y$**
4. retain and recursively extend if candidate is frequent:

if  $|T_X| \geq s$ :

$$F := F \cup \{X\}$$

$$J_X := \{(z, T_{P \cup \{z\}}) \in J \mid (z, T_z) \in J, z > y, |T_{P \cup \{z\}}| \geq s\}$$

$$\text{extend-itemset}(X, T_X, J_X)$$

# Outline

1. The Frequent Itemset Problem
2. Breadth First Search: Apriori Algorithm
3. Depth First Search: Eclat Algorithm
4. Supervised Pattern Mining

# Pattern Encodings

Patterns can be used to describe data instances/transactions:

- ▶ in this context, patterns are sometimes called **codes**,
- ▶ the list of patterns a **codebook**, and
- ▶ the representation of a transaction by **pattern indicators** as **encoding**.

$\mathcal{D} := \{X_1, \dots, X_N\}$	large transaction database
$F := \{P_1, \dots, P_K\}$	frequent patterns in $\mathcal{D}$
$X'_i = (\delta(P_k \subseteq X_i))_{k=1, \dots, K}$	representation of $X_i$ by pattern indicators

Example:

$$F := \{\{1, 3, 5\}, \{2, 6\}, \{9, 13\}\}$$

$$X := \{1, 2, 3, 4, 5, 6, 7\}$$

$$X' = (1, 1, 0)$$

# Pattern Mining as Preprocessing

Given a prediction task and

a data set  $\mathcal{D}^{\text{train}} := \{(x_1, y_1), \dots, (x_n, y_n)\} \subseteq \mathcal{P}(I) \times \mathcal{Y}$ .

Procedure:

- mine all frequent patterns  $P$  in the predictors of  $\mathcal{D}^{\text{train}}$ ,
  - e.g., using Apriori on  $\{x_1, \dots, x_n\} \subseteq \mathcal{P}(I)$  with minimum support  $s$ .
- encode predictors by  $\{x_1, \dots, x_n\}$  their pattern encodings

$$z_i := (p \subseteq x_i)_{p \in P} \in \{0, 1\}^K, \quad K := |P|$$

- learn a (linear) prediction model

$$\hat{y} : \{0, 1\}^K \rightarrow \mathcal{Y}$$

on the latent features based on

$$\mathcal{D}'^{\text{train}} := \{(z_1, y_1), \dots, (z_n, y_n)\}$$

- treat the minimum support  $s$  (and thus the number  $K$  of latent dimensions) as hyperparameter.
  - e.g., find using grid search.

# Potential Effects of Using Pattern Encodings

For transaction data / frequent itemsets:

- ▶ patterns/itemsets represent **interaction effects**:

$$\delta(\{P_1, \dots, P_L\} \subseteq X) = \prod_{\ell=1}^L \delta(P_\ell \in X)$$

- ▶ possibly **useful with linear models**
  - ▶ possibly less useful with nonlinear models that model interaction effects on their own.
- ▶ frequency used as (naive) **proxy for predictivity** of an interaction.
- ▶ minimum support  $s$  treated as hyperparameter.



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- ▶ frequency used as (naive) **proxy for predictivity** of an interaction.
- ▶ minimum support  $s$  treated as hyperparameter.

For structured data (sequences, graphs, images, text, etc.):

- ▶ a way to **extract features** from structured objects.  
(i.e., to create a vector representation that can be used with any machine learning algorithm)

# Supervised Pattern Mining

Methods that extract not just

- ▶ **frequent patterns**,
- ▶ but **predictive patterns**:

would be useful as basis for prediction.

- ▶ but e.g., correlation of a pattern with a target variable does not have the closed-downward property
  - ▶ subsets of frequent subsets are frequent,
  - ▶ but subsets of predictive subsets may not be predictive.

# Outlook

- ▶ fpGrowth
- ▶ Frequent subsequences / sequential patterns
  - ▶ Apriori can be easily adapted for sequential patterns.
  - ▶ Eclat adapted to sequential patterns: PrefixScan.
  - ▶ Additional pattern symbols: wildcards.
- ▶ Frequent subgraphs / graph patterns

# Conclusion (1/2)

- ▶ Frequent Pattern Mining searches for **frequent itemsets** in large **transaction data**, i.e., aims to find all subsets with a given **minimum support**.
  - ▶ **Association rules** can be created by simply splitting frequent itemsets.
  - ▶ As subsets of frequent sets are frequent, the result set typically is huge.
    - ▶ restrict results by looking only for **maximal frequent itemsets**.
    - ▶ rank results by other measures, e.g., **lift**.
  - ▶ Any data can be represented as transaction data (evtl. with a discretization loss).
- ▶ **Apriori** enumerates all frequent itemsets using **breadth first search**:
  - ▶ only candidates with all subsets being frequent are checked (fusing of  $k - 1$ -itemsets, **pruning**).
  - ▶ every itemset can be created just once by **sorting itemsets** and adding only larger items.
  - ▶ all  $k$ -candidates can be represented compactly in a **trie** and their support be counted efficiently in a single pass over the database.

## Conclusion (2/2)

- ▶ **Eclat** enumerates all frequent itemsets using **depth first search**:
  - ▶ only candidates with all subsets being frequent are checked (fusing of  $k - 1$ -itemsets, **pruning, traversal in reverse order**).
  - ▶ every itemset can be created just once by **sorting itemsets** and adding only larger items.
  - ▶ all candidates can be represented compactly in a **trie** and their support be counted efficiently by **intersecting itemset covers**.

# Readings

- ▶ Apriori
  - ▶ [HTFF05], ch. 14.2,
  - ▶ [AS94],[Bor03].
- ▶ Eclat
  - ▶ [ST04], [Bor03].

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