Machine Learning

B. Supervised Learning: Nonlinear Models
B.5. A First Look at Bayesian and Markov Networks

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Syllabus

Fri. 25.10.	(1)	0. Introduction
		A. Supervised Learning: Linear Models & Fundamentals
Fri. 1.11.	(2)	A.1 Linear Regression
Fri. 8.11.	(3)	A.2 Linear Classification
Fri. 15.11.	(4)	A.3 Regularization
Fri. 22.11.	(5)	A.4 High-dimensional Data
		B. Supervised Learning: Nonlinear Models
Fri. 29.11.	(6)	B.1 Nearest-Neighbor Models
Fri. 6.12.	(7)	B.2 Neural Networks
Fri. 13.12.	(8)	B.3 Decision Trees
Fri. 20.12.	(9)	B.4 Support Vector Machines
		— *Christmas Break* —
Fri. 10.1.	(10)	B.5 A First Look at Bayesian and Markov Networks
		C. Unsupervised Learning
Fri. 17.1.	(11)	C.1 Clustering
Fri. 24.1.	(12)	C.2 Dimensionality Reduction
Fri. 31.1.	(13)	C.3 Frequent Pattern Mining
Fri. 7.2.	(14)	Q&A

Outline

Outline

Joint Distribution

x_1 : the sun shines

$$\left.\begin{array}{l} p(x_1 = \text{false}) = 0.25 \\ p(x_1 = \text{true}) = 0.75 \end{array}\right\} \equiv p(x_1) = \begin{array}{|cc} \text{false} & \text{true} \\ \hline 0.25 & 0.75 \end{array} = (0.25, 0.75)$$

Joint Distribution

x_1 : the sun shines

$$\left.\begin{array}{l} p(x_1 = \text{false}) = 0.25 \\ p(x_1 = \text{true}) = 0.75 \end{array}\right\} \equiv p(x_1) = \begin{array}{|cc} \text{false} & \text{true} \\ \hline 0.25 & 0.75 \end{array} = (0.25, 0.75)$$

x_2 : it rains

$$\left.\begin{array}{l} p(x_2 = \text{false}) = 0.67 \\ p(x_2 = \text{true}) = 0.33 \end{array}\right\} \equiv p(x_2) = \begin{array}{|cc} \text{false} & \text{true} \\ \hline 0.67 & 0.33 \end{array} = (0.67, 0.33)$$

Joint Distribution

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joint distribution:

$$\left.\begin{array}{ll} p(x_1 = \text{false}, x_2 = \text{false}) & = 0.07 \\ p(x_1 = \text{false}, x_2 = \text{true}) & = 0.18 \\ p(x_1 = \text{true}, x_2 = \text{false}) & = 0.6 \\ p(x_1 = \text{true}, x_2 = \text{true}) & = 0.15 \end{array}\right\} \equiv \begin{array}{ll|ll} p(x_1, x_2) & & x_2 & \\ & & \text{false} & \text{true} \\ \hline x_1 & \text{false} & 0.07 & 0.18 \\ & \text{true} & 0.6 & 0.15 \end{array}$$

Joint Distribution

x_1 : the sun shines

$$\left.\begin{array}{l} p(x_1 = \text{false}) = 0.25 \\ p(x_1 = \text{true}) = 0.75 \end{array}\right\} \equiv p(x_1) = \begin{array}{|cc} \text{false} & \text{true} \\ \hline 0.25 & 0.75 \end{array} = (0.25, 0.75)$$

x_2 : it rains

$$\left.\begin{array}{l} p(x_2 = \text{false}) = 0.67 \\ p(x_2 = \text{true}) = 0.33 \end{array}\right\} \equiv p(x_2) = \begin{array}{|cc} \text{false} & \text{true} \\ \hline 0.67 & 0.33 \end{array} = (0.67, 0.33)$$

joint distribution:

$$p(x_1, x_2) = \begin{array}{cc|cc} & & \multicolumn{2}{c}{x_2} \\ & & \text{false} & \text{true} \\ \hline x_1 & \text{false} & 0.07 & 0.18 \\ & \text{true} & 0.6 & 0.15 \end{array} = \begin{pmatrix} 0.07 & 0.18 \\ 0.6 & 0.15 \end{pmatrix}$$

Independence

for two variables:

$$p(x, y) = p(x) \cdot p(y)$$

for two variable subsets:

$$p(x_1, x_2, \ldots, x_M) = p(x_I) \cdot p(x_J), \quad I, J \subseteq \{1, \ldots, M\}, I \cap J = \emptyset$$

Note: $x_I := \{x_{m_1}, x_{m_2}, \ldots, x_{m_K}\}$ for $I := \{m_1, m_2, \ldots, m_K\}$.

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Examples:

$$\begin{pmatrix} 0.07 & 0.18 \\ 0.6 & 0.15 \end{pmatrix}$$

not independent

$$\begin{pmatrix} 0.17 & 0.08 \\ 0.5 & 0.25 \end{pmatrix}$$

independent

Note: $x_I := \{x_{m_1}, x_{m_2}, \ldots, x_{m_K}\}$ for $I := \{m_1, m_2, \ldots, m_K\}$.

Chain Rule

$$
\begin{aligned}
p(x_1, x_2, \ldots, x_M) = & p(x_1) \\
& \cdot p(x_2 \mid x_1) \\
& \cdot p(x_3 \mid x_1, x_2) \\
& \vdots \\
& \cdot p(x_M \mid x_1, x_2, \ldots, x_{M-1})
\end{aligned}
$$

Chain Rule

$$
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& \vdots \\
& \cdot p(x_M \mid x_1, x_2, \ldots, x_{M-1})
\end{aligned}
$$

Examples:

$$
\begin{pmatrix} 0.07 & 0.18 \\ 0.6 & 0.15 \end{pmatrix} = (0.25, 0.75) \cdot \begin{pmatrix} 0.28 & 0.72 \\ 0.8 & 0.2 \end{pmatrix}
$$

Chain Rule

$$
\begin{aligned}
p(x_1, x_2, \ldots, x_M) = & p(x_1) \\
& \cdot p(x_2 \mid x_1) \\
& \cdot p(x_3 \mid x_1, x_2) \\
& \vdots \\
& \cdot p(x_M \mid x_1, x_2, \ldots, x_{M-1})
\end{aligned}
$$

Examples:

$$
\begin{pmatrix} 0.17 & 0.08 \\ 0.5 & 0.25 \end{pmatrix} = (0.25, 0.75) \cdot \begin{pmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{pmatrix}
$$

Conditional Independence

two variables x, y are **independent conditionally on variable z**:

$$x \perp y \mid z :\Leftrightarrow p(x, y \mid z) = p(x \mid z) \cdot p(y \mid z)$$

two variable sets are **independent conditionally on variables $z_1, \ldots, z_K$**:

$$\{x_1, \ldots, x_I\} \perp \{y_1, \ldots, y_J\} \mid \{z_1, \ldots, z_K\} :\Leftrightarrow$$
$$p(x_1, \ldots, x_I, y_1, \ldots, y_J \mid z_1, \ldots, z_K) = p(x_1, \ldots, x_I \mid z_1, \ldots, z_K) \cdot p(y_1, \ldots, y_J \mid z_1, \ldots, z_K)$$

Conditional Independence / Example

Example:

$$x_n \perp \{x_1, \ldots, x_{n-2}\} \mid x_{n-1} \quad \forall n \text{ (Markov property)}$$
$$\rightsquigarrow p(x_1, \ldots, x_N) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2) \cdots p(x_M \mid x_{M-1})$$

Graphical Models

- represent joint distributions of variables by graphs
  - by directed graphs: **Bayesian networks**
  - by undirected graphs: **Markov networks**
  - by mixed directed/undirected graphs.

- nodes represent random variables

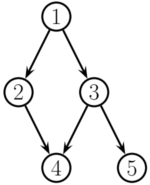
- absent edges represent conditional independence

Directed Graph Terminology

- **directed graph**: $G := (V, E), E \subseteq V \times V$
  - V set called **nodes** / **vertices**
  - E called **edges**, $(v, w) \in E$ edge from v to w.
- **adjacency matrix** $A \in \{0, 1\}^{N \times N}$

$$A_{v,w} := \delta((v, w) \in E), \quad v, w \in \{1, \ldots, N\}, N := |V|$$

- **parents**: $\text{pa}(v) := \{w \in V \mid (w, v) \in E\}$
- **children**: $\text{ch}(v) := \{w \in V \mid (v, w) \in E\}$
- **neighbors**: $\text{nbr}(v) := \text{pa}(v) \cup \text{ch}(v)$
- **family**: $\text{fam}(v) := \text{pa}(v) \cup \{v\}$
- **root**: v without parents.
- **leaf**: v without children.

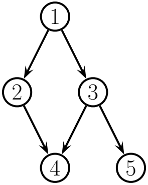


[Murphy, 2012, fig. 10.1a

Note: $\delta(P) := 1$ if proposition P is true, $:= 0$ otherwise.

Directed Graph Terminology

- **path**: $p \in V^* := \bigcup_{M\in\mathbb{N}} V^M$: $(p_m, p_{m+1}) \in E$ for all m.
  - $p = (p_1, \ldots, p_M)$, $p_m \in V$
  - **length** $|p| := M$
  - **starts at** p_1
  - **ends at** p_M
  - **paths** $G^* := \{p \in V^* \mid (p_m, p_{m+1}) \in E \quad \forall m = 1, \ldots, |p| - 1\}$.
  - $v \rightsquigarrow w$: **exists path from v to w**, i.e., $p \in G^* : p_1 = v, p_{|p|} = w$.
- **ancestors**: $\text{anc}(v) := \{w \in V \mid w \rightsquigarrow v\}$
- **descendants**: $\text{desc}(v) := \{w \in V \mid v \rightsquigarrow w\}$
- **in-degree** $|\text{pa}(v)|$
- **out-degree** $|\text{ch}(v)|$
- **degree** $|\text{nbr}(v)|$

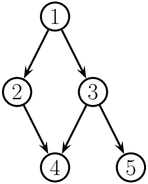


[Murphy, 2012, fig. 10.1a

Note: $V^* := \bigcup_{M\in\mathbb{N}} V^M$ **finite V-sequences**.

Directed Graph Terminology

- **cycle**/**loop** at v: $v \rightsquigarrow v$
  - **self loop**: $(v, v) \in E$
- **directed acyclic graph / DAG**:
  - directed graph without cycles.
- **topological ordering**:
  - numbering of the nodes s.t. all nodes have lower number than their children.
  - exists for DAGs.



[Murphy, 2012, fig. 10.1a

Bayesian Networks / Directed Graphical Models

A **Bayesian network** (aka **directed graphical model**) is a set of **conditional probability distributions/densities (CPDs)**

$$p(x_m \mid x_{\text{ctxt}(m)}), \quad m \in \{1, \ldots, M\}$$

s.t. the graph defined by

$$\begin{aligned} V :=& \{1, \ldots, M\} \\ E :=& \{(n, m) \mid m \in V, n \in \text{ctxt}(m)\}, \quad \text{i.e., } \text{pa}(m) := \text{ctxt}(m) \end{aligned}$$

is a DAG.

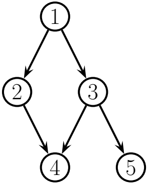
A Bayesian network defines a **factorization of the joint distribution**

$$p(x_1, \ldots, x_M) = \prod_{m=1}^{M} p(x_m \mid x_{\text{pa}(m)})$$

Bayesian Networks / Example

For the DAG below,

$$p(x_1, x_2, x_3, x_4, x_5) = p(x_1)\, p(x_2 \mid x_1)\, p(x_3 \mid x_1)\, p(x_4 \mid x_2, x_3)\, p(x_5 \mid x_3)$$



[Murphy, 2012, fig. 10.1a

Bayesian Networks / Example

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$$p(x_1, x_2, x_3, x_4, x_5) = p(x_1)\, p(x_2 \mid x_1)\, p(x_3 \mid x_1)\, p(x_4 \mid x_2, x_3)\, p(x_5 \mid x_3)$$

If

- all variables are binary and
- all CPDs given as **conditional probability tables (CPTs)**,

then the BN is defined by the following 5 CPTs:

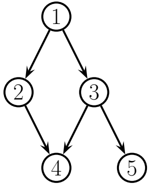
x_1	
0	...
1	...

x_2	x_1 = 0	x_1 = 1
0	...	...
1	...	...

x_3	x_1 = 0	x_1 = 1
0	...	...
1	...	...

	x_2	0		1	
	x_3	0	1	0	1
x_4	0	...	...	...	...
	1	...	...	...	...

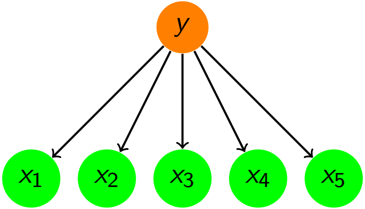
x_5	x_3 = 0	x_3 = 1
0	...	...
1	...	...



[Murphy, 2012, fig. 10.1a

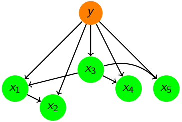
Outline

Naive Bayes Classifier

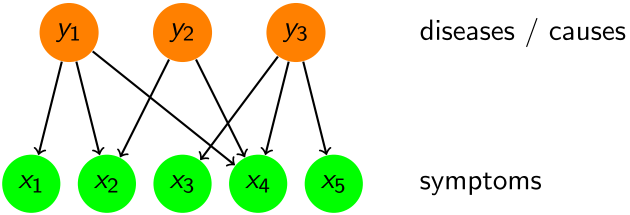


$$p(x_1, \ldots, x_M, y) = p(y)p(x_1 \mid y)p(x_2 \mid y) \cdots p(x_M \mid y)$$
$$= p(y) \prod_{m=1}^{M} p(x_m \mid y)$$

more powerful generalization:
tree-augmented naive Bayes:



Medical Diagnosis



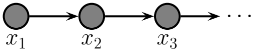
$$p(x_1, \ldots, x_M, y_1, \ldots, y_T) = \prod_{t=1}^{T} p(y_t) \prod_{m=1}^{M} p(x_m \mid y_{\text{pa}(m)})$$

- bipartite graph
- predictor variables $x_1, \ldots, x_M$ (symptoms)
- target variables $y_1, \ldots, y_T$ (diseases / causes)
  - multi-label ($\leftrightarrow$ Naive Bayes: single-label)
  - y's also could be hidden

Markov Models

first order:

$$p(x_1, \ldots, x_M) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2) \cdots p(x_M \mid x_{M-1})$$
$$= p(x_1) \prod_{m=1}^{M-1} p(x_{m+1} \mid x_m)$$

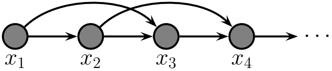


[Murphy, 2012, fig. 10.3a

Markov Models / Second Order

second order:

$$p(x_1, \ldots, x_M) = p(x_1, x_2) p(x_3 \mid x_1, x_2) p(x_4 \mid x_2, x_3) \cdots p(x_M \mid x_{M-2}, x_{M-1})$$
$$= p(x_1, x_2) \prod_{m=2}^{M-1} p(x_{m+1} \mid x_{m-1}, x_m)$$



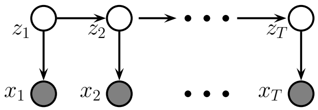
[Murphy, 2012, fig. 10.3b

Hidden Markov Models

- observed variables $x_1, \ldots, x_M$
- hidden variables $z_1, \ldots, z_M$

$$p(x_1, \ldots, x_M, z_1, \ldots, z_M) = p(z_1) \prod_{m=1}^{M-1} p(z_{m+1} \mid z_m) \prod_{m=1}^{M} p(x_m \mid z_m)$$

- **transition model** $p(z_{m+1} \mid z_m)$
- **observation model** $p(x_m \mid z_m)$



[Murphy, 2012, fig. 10.4]

Outline

The Probabilistic Inference Problem

Given

- a Bayesian network model $\theta := G = (V, E)$,
- a **query** consisting of
  - a set $X := \{x_1, \ldots, x_M\} \subseteq V$ of **predictor variables** (aka **observed**, **visible variables**)
  - with a **value** v_m for each x_m $(m = 1, \ldots, M)$ and
  - a set $Y := \{y_1, \ldots, y_T\} \subseteq V$ of **target variables** (aka **query variables**), with $X \cap Y = \emptyset$,

compute

$$p(Y \mid X = v; \theta) := p(y_1, \ldots, y_T \mid x_1 = v_1, x_2 = v_2, \ldots, x_M = v_M; \theta)$$
$$= (p(y_1 = w_1, \ldots, y_T = w_T \mid x_1 = v_1, x_2 = v_2, \ldots, x_M = v_M; \theta))_{w_1, \ldots, w_T}$$

Variables that are neither predictor variables nor target variables are called **nuisance variables**.

Inference Without Nuisance Variables

Without nuisance variables: $V = X \dot{\cup} Y$.

$$p(Y \mid X = v; \theta) \stackrel{\text{def}}{=} \frac{p(X = v, Y; \theta)}{p(X = v; \theta)} = \frac{p(X = v, Y; \theta)}{\sum_w p(X = v, Y = w; \theta)}$$

- first, clamp predictors X to their observed values v,
- then, normalize $p(X = v, Y; \theta)$ to sum to 1 (over Y).
- $p(X = v; \theta)$ **likelihood of the data** / **probability of evidence** is a constant.

Note: Summation over w is over all possible values of variables Y.

Inference With Nuisance Variables

Nuisance variables: $Z := \{z_1, \ldots, z_K\} := V \setminus (X \dot{\cup} Y)$.

1. add to target variables
2. answer resulting query without nuisance variables: $p(Y, Z \mid X)$.
3. **marginalize out** nuisance variables:

$$p(Y \mid X = v; \theta) \overset{\text{marginalization}}{=} \sum_u p(Y, Z = u \mid X = v; \theta)$$

Note: Summation over u is over all possible values of variables Z.

Inference With Nuisance Variables

Nuisance variables: $Z := \{z_1, \ldots, z_K\} := V \setminus (X \dot{\cup} Y)$.

1. add to target variables
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$$p(Y \mid X = v; \theta) \overset{\text{marginalization}}{=} \sum_u p(Y, Z = u \mid X = v; \theta)$$

Caveat: This is a naive algorithm never used in practice. See BN lecture for practically useful BN inference algorithms.

Note: Summation over u is over all possible values of variables Z.

Complexity of Inference

- for simplicity assume
  - all M predictor variables are nominal with L levels,
  - all K nuisance variables are nominal with L levels,
  - a single target variable: $Y = \{y\}, T = 1$ also nominal with L levels.
- without (Conditional) Independencies:
  - full table p requires $L^{M+K+1} - 1$ cells storage.
  - inference requires $O(L^{K+1})$ operations.
    - for each $Y = w$ sum over all L^K many $Z = u$.
- with (Conditional) Independencies / Bayesian network:
  - CPDs p require $O((M + K + 1)L^{\text{max indegree}+1})$ cells storage.
  - inference requires $O((K + 1)L^{\text{treewidth}+1})$ operations.
    - treewidth=1 for a chain!

Note: See the Bayesian networks lecture for BN inference algorithms.

Outline

Learning Bayesian Networks

- **parameter learning**: given
  - the structure of the network (graph G),
  - a regularization penalty $\text{Reg}(\theta)$ — for the parameters θ of the CPTs, and
  - data $x_1, \ldots, x_N$,

  learn the **CPTs** p.

$$\hat{\theta} := \arg\max_{\theta} \sum_{n=1}^{N} \log p(x_n; \theta) - \text{Reg}(\theta)$$

- **structure learning**: given
  - data,

  learn the **structure** G and the **CPTs** p.

Bayesian Approach

- in the Bayesian approach, parameters are also considered to be random variables, thus,
- learning is just a special type of inference (with the parameters as targets)
- information about the distribution of the parameters before seeing the data is required (**prior distribution** $p(\theta)$)
- **parameter learning**: given
  - the structure of the network (graph G) and
  - a prior distribution $p(\theta)$ of the parameters,
  - data $x_1, \ldots, x_N$,

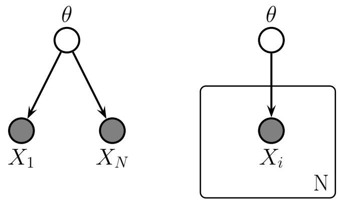
  learn the **CPTs** p.

$$\hat{\theta} := \arg\max_{\theta} \sum_{n=1}^{N} \log p(x_n; \theta) + \log p(\theta)$$

Plate Notation

- variables on plates are **duplicated**
  - the number of copies is given in the lower right corner.
- an **index** is used to differentiate copies of the same variable.

Example 1: data $x_1, \ldots, x_N$ is independently identically distributed (iid)

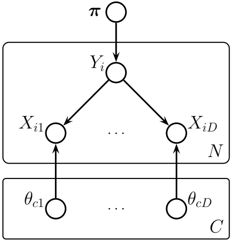


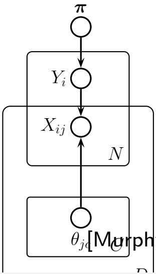
[Murphy, 2012, fig. 10.7]

Plate Notation

- variables on plates are **duplicated**
  - the number of copies is given in the lower right corner.
- an **index** is used to differentiate copies of the same variable.
- variables being in **several plates** will be duplicated for every combination, i.e., have several indices.
  - for clarity, the index should be added to the plate (but often is omitted).

Example 2: Naive Bayes classifier.





[Murphy, 2012, fig. 10.8]

Learning from Complete Data

Likelihood decomposes w.r.t. graph structure:

$$p(\mathcal{D} \mid \theta) := \prod_{n=1}^{N} p(x_n \mid \theta)$$
$$= \prod_{n=1}^{N} \prod_{m=1}^{M} p(x_{n,m} \mid x_{n,\text{pa}(m)}, \theta_m)$$
$$= \prod_{m=1}^{M} \prod_{n=1}^{N} p(x_{n,m} \mid x_{n,\text{pa}(m)}, \theta_m)$$
$$= \prod_{m=1}^{M} p(\mathcal{D}_m \mid \theta_m)$$

where θ_m are the parameters of $p(x_m \mid \text{pa}(m))$

Note: In Bayesian contexts, often $p(\ldots \mid \theta)$ is used instead of $p(\ldots ; \theta)$.

Learning from Complete Data

If the prior also factorizes,

$$p(\theta) = \prod_{m=1}^{M} p(\theta_m)$$

then the posterior factorizes as well

$$p(\theta \mid \mathcal{D}) \propto p(\mathcal{D} \mid \theta)p(\theta) = \prod_{m=1}^{M} p(\mathcal{D}_m \mid \theta_m)p(\theta_m)$$

and the parameters θ_m of each CPT can be estimated independently.

Note: In Bayesian contexts, often $p(\ldots \mid \theta)$ is used instead of $p(\ldots ; \theta)$.

Learning from Complete Data / Dirichlet Prior

If

- all variables are nominal,
- variable m has L_m levels ($m = 1, \ldots, M$), and

parameters θ of CPTs are

$$p(x_m \mid x_{\text{pa}(m)}) = \theta_{m,c,l}, \quad c := x_{\text{pa}(m)}, l := x_m$$
$$\text{with} \sum_{l=1}^{L} \theta_{m,c,l} = 1, \quad \forall m, c$$

and a **Dirichlet distribution** for each row in the CPT

$$\theta_{m,c,\cdot} \sim \text{Dir}(\alpha_{m,c}), \quad \alpha_{m,c} \in (\mathbb{R}_0^+)^{L_m}$$

is a useful prior.

Learning from Complete Data / Dirichlet Prior

Then the posterior $p(\theta_{m,c,\cdot} \mid \mathcal{D})$ is also Dirichlet:

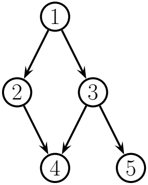
$$\theta_{m,c,\cdot} \mid \mathcal{D} \sim \text{Dir}(\alpha_{m,c} + N_{m,c})$$

$$N_{m,c,l} := \sum_{n=1}^{N} \delta(x_{n,m} = l, x_{n,\text{pa}(m)=c})$$

$$\text{with mean } \bar{\theta}_{m,c,l} = \frac{N_{m,c,l} + \alpha_{m,c,l}}{\sum_{l'=1}^{L} N_{m,c,l'} + \alpha_{m,c,l'}}$$

Learning from Complete Data / Example

graph structure:



data:

x_1	x_2	x_3	x_4	x_5
0	0	1	0	0
0	1	1	1	1
1	1	0	1	0
0	1	1	0	0
0	1	1	1	0

prior:

$$p(\theta_{m,c}) := \text{Dir}(1,1) \quad \forall m, c$$

learned parameters for CPT of x_4 ($m = 4$):

$c = x_{\text{pa}(m)}$		$N_{m,c,l}$		$\bar{\theta}_{m,c,l}$	
x_2	x_3	$N_{4,c,1}$	$N_{4,c,0}$	$\bar{\theta}_{4,c,1}$	$\bar{\theta}_{4,c,0}$
0	0	0	0	1/2	1/2
1	0	1	0	2/3	1/3
0	1	0	1	1/3	2/3
1	1	2	1	3/5	2/5

[Murphy, 2012, fig. 10.1a

Learning BN from Complete Data / Algorithm

learn-bn-params$(\mathcal{D}^{\text{train}} := \{x_1, \ldots, x_N\} \subset \mathcal{X}_1 \times \cdots \times \mathcal{X}_M, G, \alpha)$:
    for $n := 1 : N$:
        for $m := 1 : M$:
            $\alpha_{m, x_{n,m}, x_{n,\text{pa}(m)}} \mathrel{+}= 1$
    return α

where

- $\mathcal{X}_m := \{1, \ldots, L_m\}$ discrete domains of variable X_m (having L_m different levels)
- G is a DAG on $\{1, \ldots, M\}$
- $(\alpha_{m,l,c})_{m=1:M, l=1:L_m, c \in \prod_{c \in \text{pa(m)}} L_c} \geq 0$ the Dirichlet prior of the parameters

Learning with Missing and/or Hidden Variables

Learning with

- missing values or
- hidden variables

is more complicated as

- the likelihood no longer factorizes and
- neither is convex.

$\rightsquigarrow$ use iterative approximation algorithms to find a local MAP or ML optimum.

Summary

- **Bayesian Networks** define a joint probability distribution by a **factorization of conditional probability distributions (CPDs)** $p(x_n \mid \text{pa}(x_n))$
  - Conditions $\text{pa}(m)$ form a DAG.
  - For nominal variables, all CPDs can be represented as tables (CPTs).
  - Storage complexity is $O(L^{\text{max indegree}+1})$ (instead of $O(L^M)$).
- Many model classes essentially are Bayesian networks:
  - **Naive Bayes classifier**, **Markov Models**, **Hidden Markov Models**
- **Inference** in BN means to compute the (marginal joint) distribution of target variables given observed **evidence** of some predictor variables.
  - A Bayesian network can answer queries for arbitrary targets (not just a predefined one as most predictive models).
  - **Nuisance variables** (for a query) are variables neither observed nor used as targets.

Summary (2/2)

- **Learning BN** has to distinguish between
  - **parameter learning**: learn just the CPDs for a given graph, vs.
  - **structure learning**: learn both, graph and CPDs.
- Parameter learning the **maximum aposteriori (MAP)** for BN with CPTs and **Dirichlet prior** can be done simply by counting the frequencies of families in the data.

Further Readings

- [Murphy, 2012, chapter 10].

References

Kevin P. Murphy. *Machine Learning: A Probabilistic Perspective*. The MIT Press, 2012.