Machine Learning 2

D.1. Latent Dirichlet Allocation (LDA)

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Syllabus

			A. Advanced Supervised Learning
Fri.	24.4.	(1)	A.1 Generalized Linear Models
Fri.	1.5.	—	— *Labour Day* —
Fri.	8.5.	(2)	A.2 Gaussian Processes
Fri.	15.5.	(3)	A.3 Advanced Support Vector Machines
			B. Ensembles
Fri.	22.5.	(4)	B.1 Stacking
			& B.2 Boosting
Fri.	29.5.	(5)	B.3 Mixtures of Experts
Fri.	5.6.	—	— *Pentecoste Break* —
			C. Sparse Models
Fri.	12.6.	(6)	C.1 Homotopy and Least Angle Regression
Fri.	19.6.	(7)	C.2 Proximal Gradients
Fri.	26.6.	(8)	C.3 Laplace Priors
Fri.	3.7.	(9)	C.4 Automatic Relevance Determination
			D. Complex Predictors
Fri.	10.7.	(10)	D.1 Latent Dirichlet Allocation (LDA)
Fri.	17.7.	(11)	Q & A

Outline

1. The LDA Model

2. Learning LDA via Gibbs Sampling

3. Learning LDA via Collapsed Gibbs Sampling

4. Learning LDA via Variational Inference

5. Supervised LDA

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Documents / Finite Discrete Sequences

- instances $x_n \in \mathcal{A}^*$ are **discrete sequences**
  - $\mathcal{A} := \{1, \ldots, A\}$ called **dictionary** / **alphabet** ($A \in \mathbb{N}$), where $a \in A$ denotes the a-th **word** / **symbol** / **token**.
  - $\mathcal{A}^* := \bigcup_{\ell=1}^{\infty} \mathcal{A}^\ell$ called **documents** / **finite $\mathcal{A}$-sequences**.
  - $M_n := |x_n| := \ell$ called **length** (for $x_n \in \mathcal{A}^\ell$).
  - $x_{n,m}$ called **m-th word of x_n**.

- if there are no sequential effects (order does not matter), documents can be described by their **word frequencies** (**bag of words**):

$$\tilde{x}_{n,a} := |\{m \in \{1, \ldots, |x_n|\} \mid x_{n,m} = a\}|, \quad a \in \mathcal{A}$$

The LDA Model

$$
\begin{aligned}
p(x_{n,m} \mid z_{n,m} = k, \phi) &:= \mathrm{Cat}(x_{n,m} \mid \phi_k), \quad n = 1, \ldots, N, m = 1, \ldots, M_n \\
p(z_{n,m} \mid \pi_n) &:= \mathrm{Cat}(z_{n,m} \mid \pi_n), \quad n = 1, \ldots, N, m = 1, \ldots, M_n \\
p(\phi_k \mid \beta) &:= \mathrm{Dir}(\phi_k \mid \beta \, 1_A), \quad k = 1, \ldots, K \\
p(\pi_n \mid \gamma) &:= \mathrm{Dir}(\pi_n \mid \gamma \, 1_K), \quad n = 1, \ldots, N
\end{aligned}
$$

- $z_{n,m} \in \{1, \ldots, K\}$: topic the m-th word of document n belongs to.
- $\phi_k \in \Delta^A$: word probabilities of topic k.
- $\pi_n \in \Delta^K$: topic probabilities of document n.
- $\beta, \gamma \in \mathbb{R}^+$: priors of ϕ and π.

Note: $\Delta^K := \{z \in \mathbb{R}^K \mid z \geq 0, \sum_{k=1}^K z_k = 1\}$.

The LDA Model

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\begin{aligned}
p(x_{n,m} \mid z_{n,m} = k, \phi) &:= \mathrm{Cat}(x_{n,m} \mid \phi_k),\\
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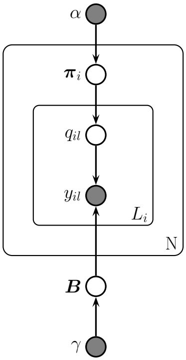
p(\phi_k \mid \beta) &:= \mathrm{Dir}(\phi_k \mid \beta\, 1_A),\\
p(\pi_n \mid \gamma) &:= \mathrm{Dir}(\pi_n \mid \gamma\, 1_K),
\end{aligned}
$$



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[Mur12, fig. 27.2]

Example $p(x_{n,m} \mid z_{n,m}, \phi)$

Topic 77

word	prob.
MUSIC	.090
DANCE	.034
SONG	.033
PLAY	.030
SING	.026
SINGING	.026
BAND	.026
PLAYED	.023
SANG	.022
SONGS	.021
DANCING	.020
PIANO	.017
PLAYING	.016
RHYTHM	.015
ALBERT	.013
MUSICAL	.013

Topic 82

word	prob.
LITERATURE	.031
POEM	.028
POETRY	.027
POET	.020
PLAYS	.019
POEMS	.019
PLAY	.015
LITERARY	.013
WRITERS	.013
DRAMA	.012
WROTE	.012
POETS	.011
WRITER	.011
SHAKESPEARE	.010
WRITTEN	.009
STAGE	.009

Topic 166

word	prob.
PLAY	.136
BALL	.129
GAME	.065
PLAYING	.042
HIT	.032
PLAYED	.031
BASEBALL	.027
GAMES	.025
BAT	.019
RUN	.019
THROW	.016
BALLS	.015
TENNIS	.011
HOME	.010
CATCH	.010
FIELD	.010

[Mur12, fig. 27.4]

Example $x_{n,m}, z_{n,m}$

Document #29795

Bix beiderbecke, at age^{060} fifteen^{207}, sat^{174} on the slope^{071} of a bluff^{055} overlooking^{027} the mississippi^{137} river^{137}. He was listening^{077} to music^{077} coming^{009} from a passing^{043} riverboat. The music^{077} had already captured^{006} his heart^{157} as well as his ear^{119}. It was jazz^{077}. Bix beiderbecke had already had music^{077} lessons^{077}. He showed^{002} promise^{134} on the piano^{077}, and his parents^{035} hoped^{268} he might consider^{118} becoming a concert^{077} pianist^{077}. But bix was interested^{268} in another kind^{050} of music^{077}. He wanted^{268} to play^{077} the cornet. And he wanted^{268} to play^{077} jazz^{077}...

Document #1883

There is a simple^{050} reason^{106} why there are so few periods^{078} of really great theater^{082} in our whole western^{046} world. Too many things^{300} have to come right at the very same time. The dramatists must have the right actors^{082}, the actors^{082} must have the right playhouses, the playhouses must have the right audiences^{082}. We must remember^{288} that plays^{082} exist^{143} to be performed^{077}, not merely^{050} to be read^{254}. (even when you read^{254} a play^{082} to yourself, try^{288} to perform^{062} it, to put^{174} it on a stage^{078}, as you go along.) as soon^{028} as a play^{082} has to be performed^{082}, then some kind^{126} of theatrical^{082}...

Document #21359

Jim^{296} has a game^{166} book^{254}. Jim^{296} reads^{254} the book^{254}. Jim^{296} sees^{081} a game^{166} for one. Jim^{296} plays^{166} the game^{166}. Jim^{296} likes^{081} the game^{166} for one. The game^{166} book^{254} helps^{081} jim^{296}. Don^{180} comes^{040} into the house^{038}. Don^{180} and jim^{296} read^{254} the game^{166} book^{254}. The boys^{020} see a game^{166} for two. The two boys^{020} play^{166} the game^{166}. The boys^{020} play^{166} the game^{166} for two. The boys^{020} like the game^{166}. Meg^{282} comes^{040} into the house^{282}. Meg^{282} and don^{180} and jim^{296} read^{254} the book^{254}. They see a game^{166} for three. Meg^{282} and don^{180} and jim^{296} play^{166} the game^{166}. They play^{166}...

[Mur12, fig. 27.5]

The LDA Model

$$
\begin{aligned}
p(x_{n,m} \mid z_{n,m} = k, \phi) &:= \mathrm{Cat}(x_{n,m} \mid \phi_k) \\
p(z_{n,m} \mid \pi_n) &:= \mathrm{Cat}(z_{n,m} \mid \pi_n) \\
p(\phi_k \mid \beta) &:= \mathrm{Dir}(\phi_k \mid \beta\, 1_A) \\
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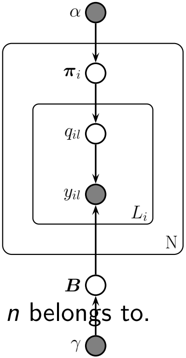

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x_7 = "In Franfurt, many banks are located at the banks of Main."
Q: How can LDA model that

- $x_{7,4}$="bank" denotes a credit institute and belongs to topic 1 "finance",
- $x_{7,9}$="bank" denotes a river bank and belongs to topic 2 "environment"?

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Learning via Parameter Sampling

The posterior

$$p(\theta \mid \mathcal{D}) \propto p(\mathcal{D} \mid \theta)$$

describes the **distribution of the parameters given the data**.
If we can **sample parameters** from this distribution

$$\theta_1, \theta_2, \ldots, \theta_S \sim p(\theta \mid \mathcal{D})$$

we can

- estimate **expected parameter values** and **their variances** from this parameter sample:

$$\hat{\theta} := E(\theta \mid \mathcal{D}) \approx \frac{1}{S} \sum_{s=1}^{S} \theta_s, \qquad V(\theta \mid \mathcal{D}) \approx \frac{1}{S-1} \sum_{s=1}^{S} (\theta_s - E(\theta \mid \mathcal{D}))^2$$

- **predict targets** for new instances x via model averaging:

$$p(y \mid x, \theta_{1:S}) = \frac{1}{S} \sum_{s=1}^{S} p(y \mid x, \theta_s)$$

Sampling

- for most closed-form distributions $p(x)$ there exist efficient sampling methods
  - categorical, normal, ...
- but most posteriors are not closed-form distributions.
  - but for example products thereof.

Gibbs Sampling

- task: sample from $p(x_1, \ldots, x_N)$
- problem:
  - assume sampling from the joint distribution $p(x_1, \ldots, x_N)$ is difficult.
  - assume sampling from marginals $p(x_n)$ or partial conditionals $p(x_n \mid \text{some } x_{n'})$ is also difficult.
  - assume sampling from all **full conditionals** $p(x_n \mid x_{-n})$ is easy.

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  - assume sampling from all **full conditionals** $p(x_n \mid x_{-n})$ is easy.

Gibbs sampling: given last sample x^s, sample x^{s+1} one variable at a time:

$$
\begin{aligned}
x_1^{s+1} &\sim p(x_1 \mid x_{2:N} = x_{2:N}^s) \\
x_2^{s+1} &\sim p(x_2 \mid x_{1:1} = x_{1:1}^{s+1}, x_{3:N} = x_{3:N}^s) \\
&\vdots \\
x_n^{s+1} &\sim p(x_n \mid x_{1:n-1} = x_{1:n-1}^{s+1}, x_{n+1:N} = x_{n+1:N}^s) \\
&\vdots \\
x_N^{s+1} &\sim p(x_N \mid x_{1:N-1} = x_{1:N-1}^{s+1})
\end{aligned}
$$

Gibbs Sampling

- the distribution created by the Gibbs sampler eventually will converge to $p(x_1, \ldots, x_N)$

- start Gibbs sampling with an arbitrary x^0
  - but ensure that $p(x^0) > 0$!
  - also consider restarts.

- throw away the first examples (**burn in**).
  - only after a while the chain has converged to the stationary distribution $p(x_1, \ldots, x_N)$.
  - typical are 100-10,000 examples

- sometimes some variables can be marginalized out, improving the performance of the Gibbs sampler (**collapsed Gibbs sampling**, **Rao-Blackwellisation**)

Gibbs Sampling for LDA

$$p(x_{n,m} \mid z_{n,m} = k, \phi) := \text{Cat}(x_{n,m} \mid \phi_k) = \phi_{k,x_{n,m}}$$
$$p(z_{n,m} \mid \pi_n) := \text{Cat}(z_{n,m} \mid \pi_n) = \pi_{n,z_{n,m}}$$
$$p(\phi_k \mid \beta) := \text{Dir}(\phi_k \mid \beta \, 1_A) \propto \prod_{a=1}^{A} \phi_{k,a}^{\beta_a - 1}$$
$$p(\pi_n \mid \gamma) := \text{Dir}(\pi_n \mid \gamma \, 1_K) \propto \prod_{k=1}^{K} \pi_{n,k}^{\gamma_k - 1}$$

Full conditionals: 1. z

$$p(z_{n,m} = k \mid \phi, \pi_n) \propto p(x_{n,m} \mid z_{n,m} = k, \phi) p(z_{n,m} = k \mid \pi_n) = \phi_{k,x_{n,m}} \pi_{n,k}$$

Gibbs Sampling for LDA

$$p(x_{n,m} \mid z_{n,m} = k, \phi) := \text{Cat}(x_{n,m} \mid \phi_k) = \phi_{k,x_{n,m}}$$

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$$p(\pi_n \mid \gamma) := \text{Dir}(\pi_n \mid \gamma\, 1_K) \propto \prod_{k=1}^{K} \pi_{n,k}^{\gamma_k - 1}$$

Full conditionals: 2. π

$$p(\pi_n \mid z_n, \phi) \propto p(\pi_n \mid \gamma) \prod_{m=1}^{M_n} p(z_{n,m} = k \mid \pi_n)$$

$$\propto \prod_{k=1}^{K} \pi_{n,k}^{\gamma_k - 1} \prod_{m=1}^{M_n} \prod_{k=1}^{K} \pi_{n,k}^{\delta(z_{n,m}=k)}$$

Q: Do you recognize this distribution?

Gibbs Sampling for LDA

$$p(x_{n,m} \mid z_{n,m} = k, \phi) := \text{Cat}(x_{n,m} \mid \phi_k) = \phi_{k,x_{n,m}}$$

$$p(z_{n,m} \mid \pi_n) := \text{Cat}(z_{n,m} \mid \pi_n) = \pi_{n,z_{n,m}}$$

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$$p(\pi_n \mid \gamma) := \text{Dir}(\pi_n \mid \gamma\, 1_K) \propto \prod_{k=1}^{K} \pi_{n,k}^{\gamma_k - 1}$$

Full conditionals: 2. π

$$p(\pi_n \mid z_n, \phi) \propto p(\pi_n \mid \gamma) \prod_{m=1}^{M_n} p(z_{n,m} = k \mid \pi_n)$$

$$\propto \prod_{k=1}^{K} \pi_{n,k}^{\gamma_k - 1} \prod_{m=1}^{M_n} \prod_{k=1}^{K} \pi_{n,k}^{\delta(z_{n,m}=k)}$$

$$= \text{Dir}((\gamma_k + \sum_{m=1}^{M} \delta(z_{n,m} = k))_{k=1:K})$$

Gibbs Sampling for LDA

$$p(x_{n,m} \mid z_{n,m} = k, \phi) := \text{Cat}(x_{n,m} \mid \phi_k) = \phi_{k,x_{n,m}}$$

$$p(z_{n,m} \mid \pi_n) := \text{Cat}(z_{n,m} \mid \pi_n) = \pi_{n,z_{n,m}}$$

$$p(\phi_k \mid \beta) := \text{Dir}(\phi_k \mid \beta \, 1_A) \propto \prod_{a=1}^{A} \phi_{k,a}^{\beta_a - 1}$$

$$p(\pi_n \mid \gamma) := \text{Dir}(\pi_n \mid \gamma \, 1_K) \propto \prod_{k=1}^{K} \pi_{n,k}^{\gamma_k - 1}$$

Full conditionals: 3. ϕ

$$p(\phi_k \mid z, \pi) \propto \Big(\prod_{n=1}^{N} \prod_{m=1}^{M_n} p(x_{n,m} = a \mid z_{n,m} = k, \phi_k) \, p(\phi_k \mid \beta)\Big)_{a=1:A}$$

$$= \text{Dir}\Big(\big(\beta_a + \sum_{n=1}^{N} \sum_{m=1}^{M} \delta(x_{n,m} = a, z_{n,m} = k)\big)_{a=1:A}\Big)$$

Gibbs Sampling for LDA

- initialize randomly

$$\pi_n \sim \text{Dir}(\gamma 1_K), \quad \phi_k \sim \text{Dir}(\beta 1_A)$$

- sample iteratively:

$$z_{n,m} \sim \text{Cat}((\phi_{k,x_{n,m}} \pi_{n,k})_{k=1:K}), \quad \forall n \forall m$$

$$\pi_n \sim \text{Dir}((\gamma_k + \sum_{m=1}^{M} \delta(z_{n,m} = k))_{k=1:K}), \quad \forall n$$

$$\phi_k \sim \text{Dir}((\beta_a + \sum_{n=1}^{N} \sum_{m=1}^{M} \delta(x_{n,m} = a, z_{n,m} = k))_{a=1:A}), \quad \forall k$$

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Counts

$$c_{n,a,k} := \sum_{m=1}^{M_n} \delta(x_{n,m} = a, z_{n,m} = k)$$

$$c_{n,k} := \sum_{a=1}^{A} c_{n,a,k}$$

$$c_{a,k} := \sum_{n=1}^{N} c_{n,a,k}$$

$$c_k := \sum_{a=1}^{A} \sum_{n=1}^{N} c_{n,a,k}$$

$$c_n := \sum_{a=1}^{A} \sum_{k=1}^{K} c_{n,a,k} = M_n$$

Marginals over π

$$\begin{aligned}
p(z_n \mid \gamma) &= \int \Big(\prod_{m=1}^{M_n} \text{Cat}(z_{n,m} \mid \pi_n)\Big)\text{Dir}(\pi_n \mid \gamma 1_K) d\pi_n \\
&= \int \prod_{k=1}^{K} \pi_{n,k}^{c_{n,k}} \frac{\Gamma(K\gamma)}{\Gamma(\gamma)^K} \pi_{n,k}^{\gamma} d\pi_n \\
&= \frac{\Gamma(K\gamma)}{\Gamma(\gamma)^K} \frac{\prod_{k=1}^K \Gamma(\gamma + c_{n,k})}{\Gamma(\sum_{k=1}^K \gamma + c_{n,k})} \int \underbrace{\frac{\Gamma(\sum_{k=1}^K \gamma + c_{n,k})}{\prod_{k=1}^K \Gamma(\gamma + c_{n,k})} \prod_{k=1}^{K} \pi_{n,k}^{\gamma + c_{n,k}}}_{=\text{Dir}(\pi_n \mid (\gamma + c_{n,k})_{k=1:K})} d\pi_n \\
&= \frac{\Gamma(K\gamma)}{\Gamma(\gamma)^K} \frac{\prod_{k=1}^K \Gamma(c_{n,k} + \gamma)}{\Gamma(M_n + K\gamma)}
\end{aligned}$$

Marginals over π and ϕ

$$p(z \mid \gamma) = \prod_{n=1}^{N} \int (\prod_{m=1}^{M_n} \text{Cat}(z_{n,m} \mid \pi_n))\text{Dir}(\pi_n \mid \gamma 1_K) d\pi_n$$

$$= \left(\frac{\Gamma(K\gamma)}{\Gamma(\gamma)^K}\right)^N \prod_{n=1}^{N} \frac{\prod_{k=1}^{K} \Gamma(c_{n,k} + \gamma)}{\Gamma(M_n + K\gamma)}$$

Marginals over π and ϕ

$$p(z \mid \gamma) = \prod_{n=1}^{N} \int (\prod_{m=1}^{M_n} \text{Cat}(z_{n,m} \mid \pi_n))\text{Dir}(\pi_n \mid \gamma 1_K) d\pi_n$$

$$= \left(\frac{\Gamma(K\gamma)}{\Gamma(\gamma)^K}\right)^N \prod_{n=1}^{N} \frac{\prod_{k=1}^{K} \Gamma(c_{n,k} + \gamma)}{\Gamma(M_n + K\gamma)}$$

$$p(x \mid z, \beta) = \prod_{k=1}^{K} \int (\prod_{(n,m):z_{n,m}=k} \text{Cat}(x_{n,m} \mid \phi_k))\text{Dir}(\phi_k \mid \beta 1_K) d\phi_k$$

$$= \left(\frac{\Gamma(A\beta)}{\Gamma(\beta)^A}\right)^K \prod_{k=1}^{K} \frac{\prod_{a=1}^{A} \Gamma(c_{a,k} + \beta)}{\Gamma(c_k + A\beta)}$$

Conditional Probability for Single $z_{n,m}$

$$p(z \mid x, \beta, \gamma) \overset{\text{Bayes}}{=} \frac{p(x \mid z, \beta, \gamma)\, p(z \mid \beta, \gamma)}{p(x \mid \beta, \gamma)} \propto p(x \mid z, \beta)\, p(z \mid \gamma)$$

$$\begin{aligned} p(z \mid x, \beta, \gamma) &= p(z_{n,m} \mid z_{-(n,m)}, x, \beta, \gamma)\, p(z_{-(n,m)} \mid x, \beta, \gamma) \\ &= p(z_{n,m} \mid z_{-(n,m)}, x, \beta, \gamma)\, p(z_{-(n,m)} \mid x_{-(n,m)}, \beta, \gamma) \end{aligned}$$

$\rightsquigarrow$

$$\begin{aligned} p(z_{n,m} \mid z_{-(n,m)}, x, \beta, \gamma) &= \frac{p(z \mid x, \beta, \gamma)}{p(z_{-(n,m)} \mid x_{-(n,m)}, \beta, \gamma)} \\ &\propto \frac{p(x \mid z, \beta)\, p(z \mid \gamma)}{p(x_{-(n,m)} \mid z_{-(n,m)}, \beta)\, p(z_{-(n,m)} \mid \gamma)} \end{aligned}$$

Conditional Probability for Single $z_{n,m}$

$$p(z_{n,m} \mid z_{-(n,m)}, x, \beta, \gamma) \propto \frac{p(x \mid z, \beta)\, p(z \mid \gamma)}{p(x_{-(n,m)} \mid z_{-(n,m)}, \beta)\, p(z_{-(n,m)} \mid \gamma)}$$

Conditional Probability for Single $z_{n,m}$

$$p(z_{n,m} \mid z_{-(n,m)}, x, \beta, \gamma) \propto \frac{p(x \mid z, \beta)\, p(z \mid \gamma)}{p(x_{-(n,m)} \mid z_{-(n,m)}, \beta)\, p(z_{-(n,m)} \mid \gamma)}$$

Now let $c^-_{n,a,k}$ be the counts for the leave-one-out sample $x_{-(n,m)}, z_{-(n,m)}$ (all but m-th word of document n).

$$c^-_{n,a,k} = \begin{cases} c_{n,a,k} - 1, & \text{for } x_{n,m} = a, z_{n,m} = k \\ c_{n,a,k}, & \text{else} \end{cases}$$

- all terms other than for $x_{n,m} = a, z_{n,m} = k$ cancel out.
- terms for $x_{n,m} = a, z_{n,m} = k$ can be simplified via $\Gamma(x+1)/\Gamma(x) = x$

Conditional Probability for Single $z_{n,m}$

$$p(z_{n,m} \mid z_{-(n,m)}, x, \beta, \gamma) \propto \frac{p(x \mid z, \beta)\, p(z \mid \gamma)}{p(x_{-(n,m)} \mid z_{-(n,m)}, \beta)\, p(z_{-(n,m)} \mid \gamma)}$$

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- terms for $x_{n,m} = a, z_{n,m} = k$ can be simplified via $\Gamma(x+1)/\Gamma(x) = x$

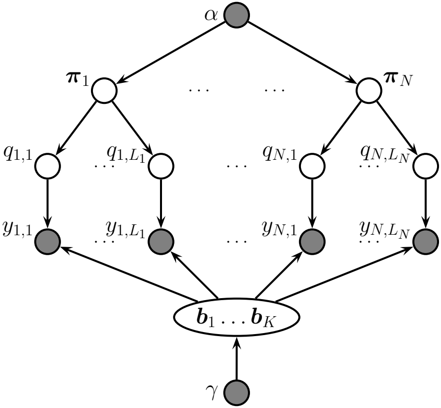
$$p(z_{n,m} = k \mid z_{-(n,m)}, x, \beta, \gamma) \propto \frac{c^-_{x_{n,m},k} + \beta}{c^-_k + A\beta} \frac{c^-_{n,k} + \gamma}{M_n + K\gamma}$$

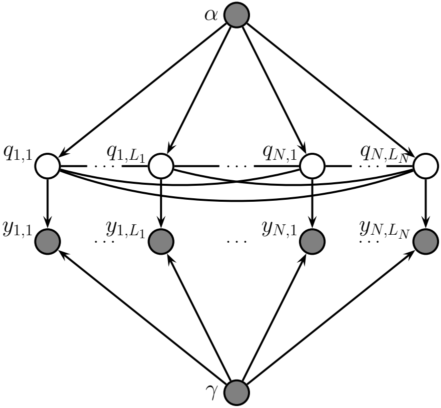
Collapsed LDA Implementation

- assign all $z_{n,m}$ randomly
- compute $c_{n,a,k}$
- for $s := 1, \ldots, S$:
  - for $n := 1, \ldots, N, \quad m := 1, \ldots, M_n$:

$$
\begin{aligned}
c_{x_{n,m}, z_{n,m}} &:= c_{x_{n,m}, z_{n,m}} - 1 \\
c_{n, z_{n,m}} &:= c_{n, z_{n,m}} - 1 \\
c_{z_{n,m}} &:= c_{z_{n,m}} - 1 \\
z_{n,m} &\sim \text{Cat}\left(\left(\frac{c^-_{x_{n,m},k} + \beta}{c^-_k + A\beta} \frac{c^-_{n,k} + \gamma}{M_n + K\gamma} \right)_{k=1:K} \right) \\
c_{x_{n,m}, z_{n,m}} &:= c_{x_{n,m}, z_{n,m}} + 1 \\
c_{n, z_{n,m}} &:= c_{n, z_{n,m}} + 1 \\
c_{z_{n,m}} &:= c_{z_{n,m}} + 1
\end{aligned}
$$

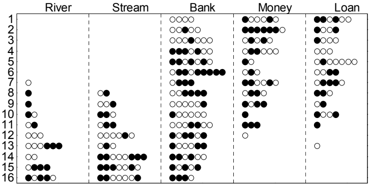
LDA vs Collapsed LDA

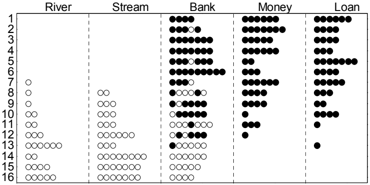




[Mur12, fig. 27.7]

Collapsed LDA / Example





$N = 16$ (rows), $A = 5$ (columns), $K = 2$ (colors)

[Mur12, fig. 27.8]

Outline

1. The LDA Model

2. Learning LDA via Gibbs Sampling

3. Learning LDA via Collapsed Gibbs Sampling

4. Learning LDA via Variational Inference

5. Supervised LDA

Variational Inference via Mean Field Approximation

To solve the inference problem

$$\text{compute } p(x_1, \ldots, x_N)$$

for intractable p, **approximate p** with a **fully factorized** density q

$$p(x_1, \ldots, x_N) \approx q(x_1, \ldots, x_N \mid \theta) := \prod_{n=1}^{N} q_n(x_n \mid \theta_n)$$

A good approximation should minimize the KL divergence of p and q:

$$(\theta_1, \ldots, \theta_N) := \arg\min_{\theta_1, \ldots, \theta_N} \mathrm{KL}(q||p)$$

$$\mathrm{KL}(q||p) := E_x(\; q(x) \log \frac{q(x)}{p(x)} \;)$$

which can be solved via coordinate descent:

$$\log q_n(x_n \mid \theta_n) = E_{x_{-n} \sim q_{-n}}(\tilde{p}(x_1, \ldots, x_N)) + \text{const}$$

where $\tilde{p}$ can be an unnormalized version of p.

Learning LDA via Mean Field Approximation

Mean field approximation

$$q(\pi_n \mid \tilde{\pi}_n) := \text{Dir}(\pi_n \mid \tilde{\pi}_n)$$
$$q(z_{n,m} \mid \tilde{z}_{n,m}) := \text{Cat}(z_{n,m} \mid \tilde{z}_{n,m})$$

in the E-step of EM leads to

E-step:

$$\tilde{z}_{n,m,k} = \phi_{x_{n,m},k} e^{\Psi(\tilde{\pi}_{n,k}) - \Psi(\sum_{k'} \tilde{\pi}_{n,k'})}$$
$$\tilde{\pi}_{n,k} = \gamma + \sum_m \tilde{z}_{n,m,k}$$

M-step:

$$\phi_{a,k} = \beta + \sum_n \sum_m \tilde{z}_{n,m,k} \delta(x_{n,m} = a)$$

Note: $E_{\pi_{n,k} \sim \text{Dir}(\tilde{\pi}_{n,k})}(\log \pi_{n,k}) = \Psi(\tilde{\pi}_{n,k}) - \Psi(\sum_{k'} \tilde{\pi}_{n,k'})$ with Ψ the digamma function.

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Adding Further Information

- Add observed class information

$$y_n \in \mathcal{Y} := \{1, \ldots, T\}, \quad n \in \{1, \ldots, N\}$$

- goal now is either
  - to analyze x_n with an LDA model and predict targets y_n based on this analysis (supervised learning) or
  - to find topics that explain both, documents x_n and their classes y_n (unsupervised learning).
- Sometimes richer information is added, e.g., images.

Joint LDA and Logistic Regression

$$p(y_n \mid \pi_n, \theta) := \text{Cat}(y_n \mid \text{logistic}(\theta^T \pi_n))$$
$$p(\theta \mid \sigma^2) := \mathcal{N}(\theta \mid 0, \sigma^2)$$

$$\begin{aligned}
p(x_{n,m} \mid z_{n,m} = k, \phi) &:= \text{Cat}(x_{n,m} \mid \phi_k), \quad n = 1, \ldots, N, m = 1, \ldots, M_n \\
p(z_{n,m} \mid \pi_n) &:= \text{Cat}(z_{n,m} \mid \pi_n), \quad n = 1, \ldots, N, m = 1, \ldots, M_n \\
p(\phi_k \mid \beta) &:= \text{Dir}(\phi_k \mid \beta \, 1_A), \quad k = 1, \ldots, K \\
p(\pi_n \mid \gamma) &:= \text{Dir}(\pi_n \mid \gamma \, 1_K), \quad n = 1, \ldots, N
\end{aligned}$$

Generative Supervised LDA

$$p(y_n \mid \bar{\pi}_n, \theta) := \text{Cat}(y_n \mid \text{logistic}(\theta^T \bar{\pi}_n)), \quad \bar{\pi}_{n,k} := \frac{1}{M_n} \sum_{m=1}^{M_n} \delta(z_{n,m} = k)$$

$$\begin{aligned}
p(x_{n,m} \mid z_{n,m} = k, \phi) &:= \text{Cat}(x_{n,m} \mid \phi_k), \quad n = 1, \ldots, N, m = 1, \ldots, M_n \\
p(z_{n,m} \mid \pi_n) &:= \text{Cat}(z_{n,m} \mid \pi_n), \quad n = 1, \ldots, N, m = 1, \ldots, M_n \\
p(\phi_k \mid \beta) &:= \text{Dir}(\phi_k \mid \beta \, 1_A), \quad k = 1, \ldots, K \\
p(\pi_n \mid \gamma) &:= \text{Dir}(\pi_n \mid \gamma \, 1_K), \quad n = 1, \ldots, N
\end{aligned}$$

Discriminative Supervised LDA

$$
\begin{aligned}
p(x_{n,m} \mid z_{n,m} = k, \phi) &:= \text{Cat}(x_{n,m} \mid \phi_k), \quad n = 1, \dots, N, m = 1, \dots, M_n \\
p(z_{n,m} \mid \pi_n, y_n = t) &:= \text{Cat}(z_{n,m} \mid A_t \pi_n), \quad n = 1, \dots, N, m = 1, \dots, M_n \\
p(\phi_k \mid \beta) &:= \text{Dir}(\phi_k \mid \beta \, 1_A), \quad k = 1, \dots, K \\
p(\pi_n \mid \gamma) &:= \text{Dir}(\pi_n \mid \gamma \, 1_K), \quad n = 1, \dots, N
\end{aligned}
$$

- $A_t \in \mathbb{R}^{K \times K}$ stochastic $(t = 1, \dots, T)$

Summary

- **Latent Dirichlet Allocation** (**LDA**) solves a clustering problem for sequence data (really: histograms)
  - clusters are called **topics**.
  - topics are described by **word/symbol probabilities**.
  - documents/sequences by **topic probabilities**.
  - a **latent variable "word topic"** for each word/element of each sequence.
    - semantically: disambiguation of the word (w.r.t. its topic)

- LDA can be learned via **Gibbs sampling**:
  - re-sample single variables from their full conditionals on all others in a round-robin fashion.
  - leads to sampling from categorical and Dirichlet distributions.

- LDA can be learned via **collapsed Gibbs sampling**:
  - integrate out word and document probabilities, leaving just the latent word topics.
  - leads to a way faster sampling from a categorical distribution only.

Summary (2/2)

- LDA can be learned via **variational inference** using **mean field approximation**.
  - approximate a distribution by a fully factorized distribution.
    - here: the distribution of the latent word topics and topic probabilities in the E-step of an EM algorithm for LDA.
    - leads to closed-form reestimation formulas.

- LDA can be extended different ways to take **document labels/ classes** into account.
  - yielding a model for text classification.
  - joint LDA and logistic regression, generative supervised LDA, discriminative supervised LDA

Further Readings

- LDA:
  - [Mur12, chapter 27.3],
- Supervised LDA and other extensions:
  - [Mur12, chapter 27.4],

References

Kevin P. Murphy.
Machine learning: a probabilistic perspective.
The MIT Press, 2012.