Modern Optimization Techniques

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Overview

Outline

1. Optimization Problems
2. Application Areas
3. Classification of optimization problems
4. Overview of the Lecture
5. Organizational Stuff

Outline

Optimization Problems

An **optimization problem** has the form:

$$\text{minimize} \quad f_0(x)$$

Where:

- $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$
- An optimal x^* exists and $f_0(x^*) = p^*$

Optimization Problems - A simple example

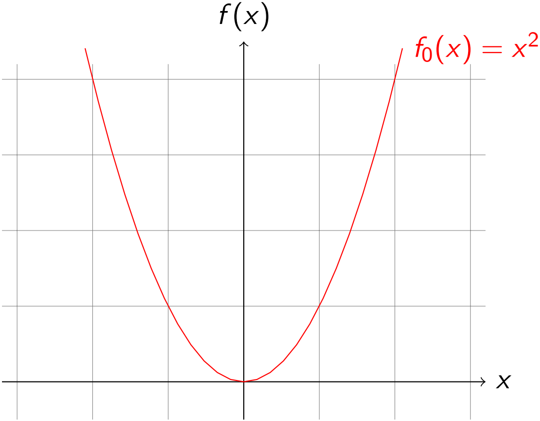
Say we have $f_0(x) = x^2$:

$$\text{minimize} \quad x^2$$

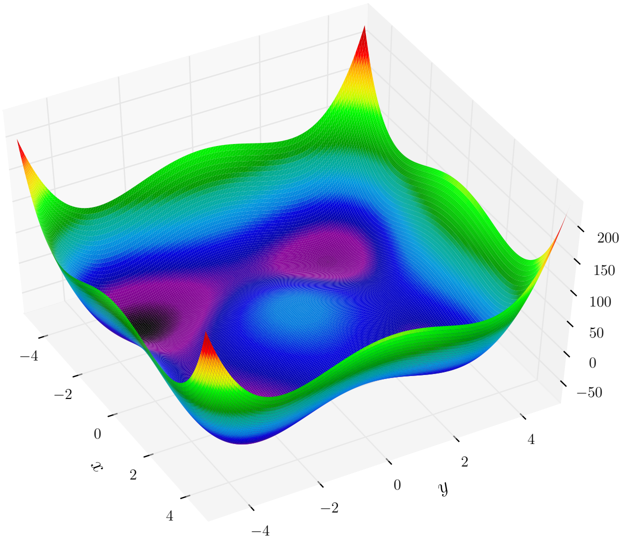
$$\frac{df_0(x)}{dx} = 0$$
$$2x = 0$$
$$x = 0$$

So:

$$x^* = 0$$
$$p^* = f_0(x^*) = 0^2 = 0$$



Optimization Problems



Optimization Problems - Constraints

A **constrained optimization problem** has the form:

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

Where:

- $f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$
- $A \in \mathbb{R}^{l \times n}$, with rank $A = l < n$
- An optimal x^* exists and $f_0(x^*) = p^*$

Optimization Problems - Vocabulary

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{aligned}
$$

Where:

- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the **objective function**
- $x \in \mathbb{R}^n$ is the optimization variable
- $(f_i)_{i=1,\ldots,m} : \mathbb{R}^n \to \mathbb{R}$ are the constraint functions

Outline

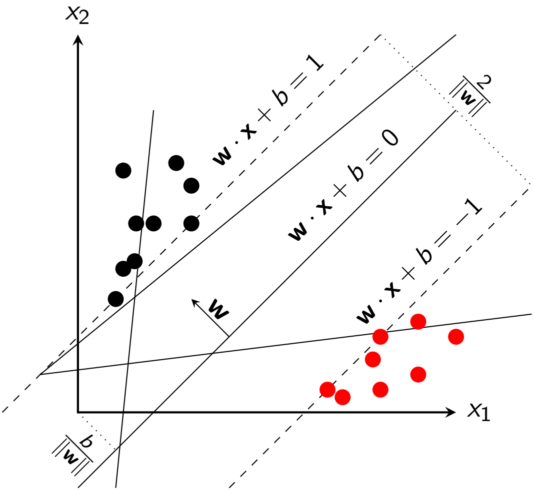
What is optimization good for?

The optimization problem is an abstraction of the problem of making the best possible choice of a vector in $\mathbb{R}^n$ from a set of candidate choices

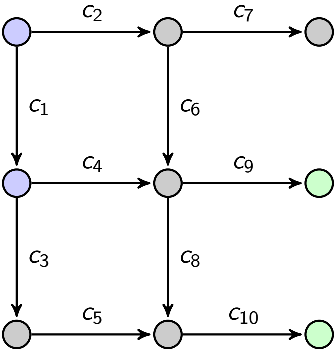
- Machine Learning
- Logistics
- Computer Vision
- Decision Making
- Device Sizing
- Scheduling
- ...

Application Areas - Machine Learning

Task: Classification



Application Areas - Logistics



Suppose we have:

- Factories
- Warehouses
- Roads with costs associated to them

Determine how many products to ship from each factory to each warehouse to minimize shipping cost while meeting warehouse demands and not exceeding factory supplies

Application Areas - Computer Vision

Outline

Classification

There are many different ways to group mathematical optimization problems.

The most common are:

- Convex vs. Non-convex
- Linear vs. Non-linear
- Constrained vs. Unconstrained

Convex Functions

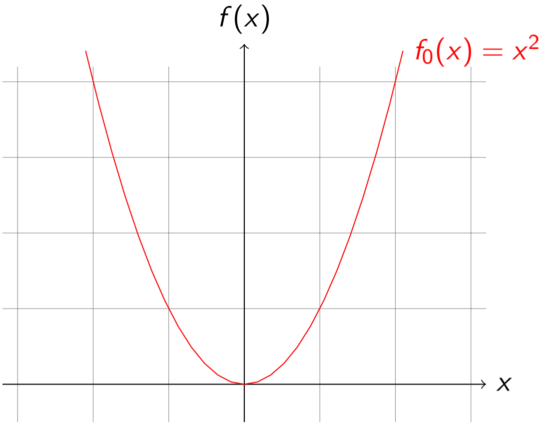
A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if it satistfies

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

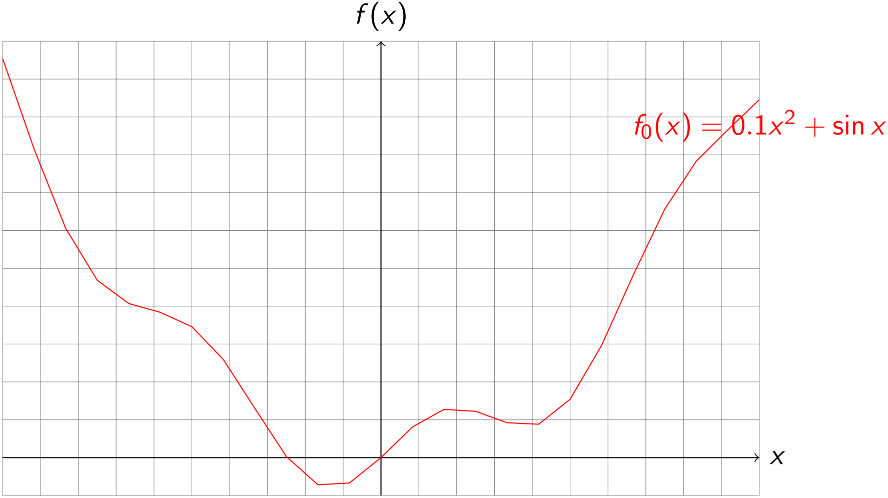
Where:

- $x, y \in \mathbb{R}^n$
- $\alpha, \beta \in \mathbb{R}$
- $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

A convex function



A non-convex function



Convex Optimization Problem

An **optimization problem**

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

is said to be convex if $f_0, \ldots f_m$ are convex

Linear and Non-Linear Problems

A function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if it satistfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

An **optimization problem** is said to be linear if the objective function f_0 and the constraints $f_1, \ldots f_m$ are also linear

Constrained and Unconstrained Problems

An **unconstrained optimization problem** has only the objective function f_0

A **constrained optimization problem** has besides objective function f_0 the constraint functions $f_1, \dots f_m$
The constraints can be formulated as

- equalities
- inequalities

Outline

Overview of the Lecture

0. **Overview**
1. **Theory**
2. **Unconstrained Optimization**
3. **Equality Constrained Methods**
4. **Inequality Constrained Methods**
5. **Non-Linear Optimization**
6. **Non-Convex Optimization**

Overview of the Lecture

0. **Overview**
1. **Theory**
   - 1.1 Convex Sets
   - 1.2 Convex Functions
   - 1.3 Convex Optimization Problems
2. **Unconstrained Optimization**
   - 2.1 Line search and Gradient Descent
   - 2.2 Newton Method
   - 2.3 Coordinate Descent
   - 2.4 Conjugate Gradient
   - 2.5 Stochastic Gradient Descent
   - 2.6 Quasi-Newton Methods

Overview of the Lecture

3. **Equality Constrained Methods**
   - 3.1 Duality
   - 3.2 Newton Methods for Equality Constrained Optimization
   - 3.3 Infeasible Start Newton Methods
4. **Inequality Constrained Methods**
   - 4.1 Interior Point Methods
   - 4.2 Barrier Methods
   - 4.3 Penalty Methods
   - 4.4 Cutting Plane Methods
5. **Non-Linear Optimization**
6. **Non-Convex Optimization**

Unconstrained Optimization Problems

An **unconstrained optimization problem** has the form:

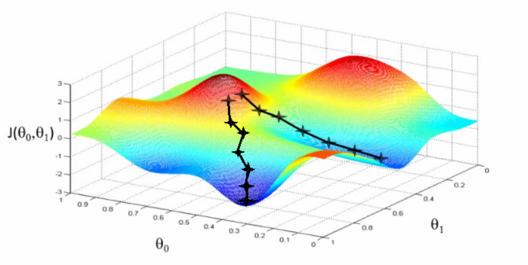
$$\text{minimize} \quad f_0(x)$$

Where:

- $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$
- An optimal x^* exists and $f_0(x^*) = p^*$

Gradient Descent

procedure GradientDescent
input: λ
    Initialize $\mathbf{x}$
    repeat
        $\mathbf{x} := \mathbf{x} - \lambda \nabla f_0(\mathbf{x})$
    until convergence
    return x
end procedure



Newton Method

procedure Newton Method
input: λ
  Initialize $\mathbf{x}$
  repeat
    $\Delta_{\mathbf{x}} := -\nabla^2 f_0(\mathbf{x})^{-1} \nabla f_0(\mathbf{x})$
    Choose step-size λ through line search
    $\mathbf{x} := \mathbf{x} + \lambda \Delta_{\mathbf{x}}$
  until convergence
  return $\mathbf{x}$
end procedure

Equality Constrained Minimization Problems

A problem of the form:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

Where:

- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is **convex** and **twice differentiable**
- $A \in \mathbb{R}^{l \times n}$, with rank $A = l < n$
- An optimal x^* exists and $f_0(x^*) = p^*$

Methods for Equality Constrained Problems

- Reformulate the problem by eliminating the constraints
- Solve the new unconstrained problem
- Convert the solution of the unconstrained problem to the constrained problem

Methods:

- Newton Method for Equality Constrained Problems
- Infeasible Start Newton

Inequality Constrained Minimization (ICM) Problems

A problem of the form:

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\
& Ax = b
\end{aligned}
$$

Where:

- $f_0, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are **convex** and **twice differentiable**
- $A \in \mathbb{R}^{l \times n}$, with rank $A = l < n$
- An optimal x^* exists and $f_0(x^*) = p^*$

Interior-point Methods

Interior Point Methods solve inequality constrained minimization problems by

1. Reducing them to a sequence of linear equality constrained problems
2. Applying Newton's method to the approximation

The Barrier Method - Algorithm

procedure BARRIER METHOD
input: strictly feasible $x^{(0)}$, $t^0 > 0$, step size $\mu > 1$, tolerance $\epsilon > 0$

$t := t^0$
$x := x^0$
while $m/t < \epsilon$ **do**
/ Centering Step */*
$x^*(t) := \arg\min_{x(t)} tf_0(x(t)) + \phi(x(t))$,
subject to $Ax(t) = b$,
starting at $x(t) = x$
$x := x^*(t)$A problem of the form:

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) = 0, \quad i = 1, \ldots, m
\end{array}
$$

Outline

Exercises and tutorials

- There wil be a weekly sheet with two exercises
- Exercises will be corrected
- Tutorials each Tuesday 14-16,
  1st tutorial at Tuesday 28.10.
- Successful participation in the tutorial gives up to 10% bonus points for the exam.

Exams and credit points

- There will be a written exam at the end of the term (2h, 4 problems).
- The course gives 6 ECTS
- The course can be used in
  - IMIT MSc. / Informatik / Gebiet KI & ML
  - Wirtschaftsinformatik MSc / Informatik / Gebiet KI & ML

Some books

- Stephen Boyd and Lieven Vandenberghe. Convex Optimization. Cambridge Univ Press, 2004.
- Suvrit Sra, Sebastian Nowozin and Stephen J. Wright. Optimization for Machine Learning. MIT Press, 2011.
- Igor Griva. Linear and nonlinear optimization. Society for Industrial and Applied Mathematics, 2009.