Modern Optimization Techniques

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Overview
Outline

1. Optimization Problems

2. Application Areas

3. Classification of optimization problems

4. Overview of the Lecture

5. Organizational Stuff
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Optimization Problems

An optimization problem has the form:

\[
\text{minimize } f_0(x)
\]

Where:

- \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \)
- An optimal \( x^* \) exists and \( f_0(x^*) = p^* \)
Optimization Problems - A simple example

Say we have \( f_0(x) = x^2 \):

\[
\text{minimize} \quad x^2
\]

\[
\frac{df_0(x)}{dx} = 0
\]

\[
2x = 0
\]

\[
x = 0
\]

So:

\[
x^* = 0
\]

\[
p^* = f_0(x^*) = 0^2 = 0
\]
Optimization Problems
A **constrained optimization problem** has the form:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

Where:

- \( f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \)
- \( A \in \mathbb{R}^{l \times n}, \) with rank \( A = l < n \)
- An optimal \( x^* \) exists and \( f_0(x^*) = p^* \)
Optimization Problems - Vocabulary

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

Where:

- \( f_0 : \mathbb{R}^n \to \mathbb{R} \) is the **objective function**
- \( x \in \mathbb{R}^n \) is the optimization variable
- \( (f_i)_{i=1,\ldots,m} : \mathbb{R}^n \to \mathbb{R} \) are the constraint functions
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What is optimization good for?

The optimization problem is an abstraction of the problem of making the best possible choice of a vector in $\mathbb{R}^n$ from a set of candidate choices.

- Machine Learning
- Logistics
- Computer Vision
- Decision Making
- Device Sizing
- Scheduling
- ...

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Application Areas - Machine Learning

Task: Classification
Suppose we have:

- Factories
- Warehouses
- Roads with costs associated to them

Determine how many products to ship from each factory to each warehouse to minimize shipping cost while meeting warehouse demands and not exceeding factory supplies.
Application Areas - Computer Vision

360° Velodyne Laserscanner
Stereo Camera Rig
Monochrome Color
GPS

AnnieWay
KIT

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Classification

There are many different ways to group mathematical optimization problems. The most common are:

- Convex vs. Non-convex
- Linear vs. Non-linear
- Constrained vs. Unconstrained
Convex Functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if it satisfies

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

Where:

- $x, y \in \mathbb{R}^n$
- $\alpha, \beta \in \mathbb{R}$
- $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$
A convex function

\[ f_0(x) = x^2 \]
A non-convex function

\[ f_0(x) = 0.1x^2 + \sin x \]
Convex Optimization Problem

An optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

is said to be convex if \( f_0, \ldots f_m \) are convex
Linear and Non-Linear Problems

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear if it satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

An **optimization problem** is said to be linear if the objective function $f_0$ and the constraints $f_1, \ldots, f_m$ are also linear.
Constrained and Unconstrained Problems

An **unconstrained optimization problem** has only the objective function $f_0$

A **constrained optimization problem** has besides objective function $f_0$ the constraint functions $f_1, \ldots, f_m$

The constraints can be formulated as

- equalities
- inequalities
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Overview of the Lecture

0. Overview
1. Theory
2. Unconstrained Optimization
3. Equality Constrained Methods
4. Inequality Constrained Methods
5. Non-Linear Optimization
6. Non-Convex Optimization
Overview of the Lecture

0. Overview

1. Theory
   ▶ 1.1 Convex Sets
   ▶ 1.2 Convex Functions
   ▶ 1.3 Convex Optimization Problems

2. Unconstrained Optimization
   ▶ 2.1 Line search and Gradient Descent
   ▶ 2.2 Newton Method
   ▶ 2.3 Coordinate Descent
   ▶ 2.4 Conjugate Gradient
   ▶ 2.5 Stochastic Gradient Descent
   ▶ 2.6 Quasi-Newton Methods
Overview of the Lecture

3. **Equality Constrained Methods**
   - 3.1 Duality
   - 3.2 Newton Methods for Equality Constrained Optimization
   - 3.3 Infeasible Start Newton Methods

4. **Inequality Constrained Methods**
   - 4.1 Interior Point Methods
   - 4.2 Barrier Methods
   - 4.3 Penalty Methods
   - 4.4 Cutting Plane Methods

5. **Non-Linear Optimization**

6. **Non-Convex Optimization**
Unconstrained Optimization Problems

An **unconstrained optimization problem** has the form:

\[
\text{minimize } f_0(x)
\]

Where:

- \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \)
- An optimal \( x^\ast \) exists and \( f_0(x^\ast) = p^\ast \)
Gradient Descent

1: procedure 
   \text{GradientDescent} 
   \textbf{input: } \lambda 
2: Initialize x 
3: repeat 
4: \hspace{1em} x := x - \lambda \nabla f_0(x) 
5: until convergence 
6: return x 
7: end procedure
Newton Method

1: procedure Newton Method
   input: \( \lambda \)
2:     Initialize \( x \)
3:     repeat
4:       \( \Delta x := -\nabla^2 f_0(x)^{-1} \nabla f_0(x) \)
5:     Choose step-size \( \lambda \) through line search
6:     \( x := x + \lambda \Delta x \)
7:     until convergence
8:     return \( x \)
9: end procedure
Equality Constrained Minimization Problems

A problem of the form:

\[
\text{minimize} \quad f_0(x) \\
\text{subject to} \quad Ax = b
\]

Where:

- \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) is \textbf{convex} and \textbf{twice differentiable}
- \( A \in \mathbb{R}^{l \times n} \), with rank \( A = l < n \)
- An optimal \( x^* \) exists and \( f_0(x^*) = p^* \)
Methods for Equality Constrained Problems

- Reformulate the problem by eliminating the constraints
- Solve the new unconstrained problem
- Convert the solution of the unconstrained problem to the constrained problem

Methods:
- Newton Method for Equality Constrained Problems
- Infeasible Start Newton
Inequality Constrained Minimization (ICM) Problems

A problem of the form:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

Where:

- \( f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \) are convex and twice differentiable
- \( A \in \mathbb{R}^{l \times n} \), with rank \( A = l < n \)
- An optimal \( x^* \) exists and \( f_0(x^*) = p^* \)
Interior Point Methods solve inequality constrained minimization problems by

1. Reducing them to a sequence of linear equality constrained problems
2. Applying Newton’s method to the approximation
The Barrier Method - Algorithm

1: procedure Barrier Method
   input: strictly feasible $x^{(0)}$, $t^0 > 0$, step size $\mu > 1$, tolerance $\epsilon > 0$

2: \hspace{1em} $t := t^0$

3: \hspace{1em} $x := x^0$

4: while $m/t < \epsilon$ do
   /* Centering Step */
   5: \hspace{1em} $x^*(t) := \text{arg min}_{x(t)} tf_0(x(t)) + \phi(x(t))$, \\
      \hspace{1em} subject to $Ax(t) = b$, \\
      \hspace{1em} starting at $x(t) = x$
   6: \hspace{1em} $x := x^*(t)$A problem of the form:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) = 0, \quad i = 1, \ldots, m
\end{align*}
\]
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Exercises and tutorials

- There will be a weekly sheet with two exercises
- Exercises will be corrected
- Tutorials each Tuesday 14-16,
  1st tutorial at Tuesday 28.10.
- Successful participation in the tutorial gives up to 10% bonus points for the exam.
Exams and credit points

- There will be a written exam at the end of the term (2h, 4 problems).
- The course gives 6 ECTS
- The course can be used in
  - IMIT MSc. / Informatik / Gebiet KI & ML
  - Wirtschaftsinformatik MSc / Informatik / Gebiet KI & ML
Some books