Modern Optimization Techniques

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Cutting Plane Methods

Outline

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Inequality Constrained Minimization (ICM) Problems

A problem of the form:

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & A\mathbf{x} = \mathbf{b} \end{aligned}$$

Where:

- $f_0, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are **convex** and **twice differentiable**
- $A \in \mathbb{R}^{q \times n}$ and $\mathbf{b} \in \mathbb{R}^q$
- A feasible optimal $\mathbf{x}^*$ exists and $f_0(\mathbf{x}^*) = p^*$

KKT Conditions

Assume that a strictly feasible solution $\mathbf{x}^*$ to the problem exists, the KKT Conditions are:

1. Primal feasibility: $f_i(\mathbf{x}^*) \leq 0$ for all i and $A\mathbf{x}^* = \mathbf{b}$
2. Dual feasibility: $\lambda \succeq 0$
3. Complementary Slackness: $\lambda_i f_i(\mathbf{x}^*) = 0$ for all i
4. Stationarity: $\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\mathbf{x}^*) + \sum_{i=1}^{q} \nu_i \nabla h_i(\mathbf{x}^*) = 0$

Outline

Cutting Plane Methods

We have seen how to solve inequality constrained problems using interior point methods

Interior point methods assume $\{f_i(\mathbf{x})\}_{i=0,\ldots,m}$ to be *convex* and *twice differentiable*

What to do if f_i is nondifferentiable?

Cutting plane methods:

- Are able to handle nondifferentiable convex problems
- Can also be applied to unconstrained minimization problems
- Require the computation of a subgradient per step
- Can be much faster than subgradient methods

Cutting Plane Methods - Basic Idea

Let us denote by $\mathcal{B} \subseteq \mathbb{R}^n$ the set of all solutions $\mathbf{x}^*$ to our problem:

$$\mathcal{B} := \{\mathbf{x}^* | f_0(\mathbf{x}^*) = p^* \wedge A\mathbf{x}^* = \mathbf{b} \wedge f_i(\mathbf{x}^*) \leq 0\}$$

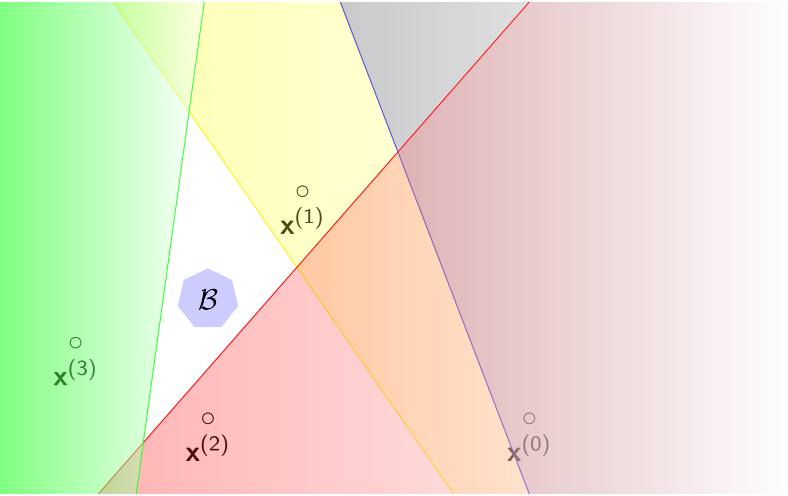
Assume we have an **oracle** who can "answer" $\mathbf{x} \overset{?}{\in} \mathcal{B}$

The oracle returns a plane that separates $\mathbf{x}$ from $\mathcal{B}$

A cutting plane method starts with an initial solution $\mathbf{x}^t$ and then:

1. Query the oracle $\mathbf{x}^t \overset{?}{\in} \mathcal{B}$
2. If $\mathbf{x}^t \in \mathcal{B}$ then stop and return $\mathbf{x}^t$
3. Generate a new point $\mathbf{x}^{t+1}$ on the other side of the plane returned by the oracle
4. Go back to step 1

Cutting Plane Methods - Basic Idea



Outline

Cutting Plane Oracle

Goal: Determine if $\mathbf{x} \overset{?}{\in} \mathcal{B}$

There are two possible outcomes of a query to the oracle:

- A positive answer if $\mathbf{x} \in \mathcal{B}$
- If $\mathbf{x} \notin \mathcal{B}$ it returns a separating hyperplane $(\mathbf{u}, v)$ between $\mathbf{x}$ and $\mathcal{B}$:

$$\mathbf{u}^T\mathbf{x}^* \leq v \quad \text{for} \quad \mathbf{x}^* \in \mathcal{B}$$
$$\mathbf{u}^T\mathbf{x} \geq v$$

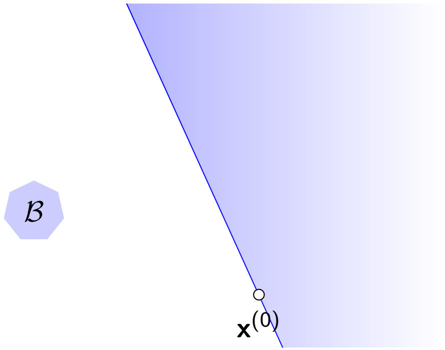
with $\mathbf{u} \in \mathbb{R}^n$ and $v \in \mathbb{R}$

This means we can eliminate (cut) all points in the halfspace $\{\alpha | \mathbf{u}^T\alpha > v\}$ from our search

Neutral cuts

If **x** is on the boundary of the halfspace the cut is called **neutral**:

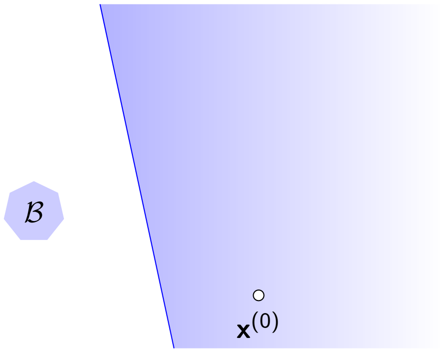
$$\mathbf{u}^T\mathbf{x} = v$$



Deep cuts

If **x** is in the interior of the halfspace that is cut we have a **deep** cut:

$$\mathbf{u}^T\mathbf{x} > v$$



Oracle for an Unconstrained Minimization Problem

For a convex $f_0 : \mathbb{R}^n \to \mathbb{R}$ and the minimization problem min f_0:

We can implement the oracle through the subdifferential $\partial f_0(\mathbf{x})$:

Be $\mathbf{g} \in \partial f_0(\mathbf{x})$, we know from the subgradient definition:

$$f_0(\alpha) \geq f_0(\mathbf{x}) + \mathbf{g}^T(\alpha - \mathbf{x})$$

thus if

$$\mathbf{g}^T(\alpha - \mathbf{x}) > 0$$

then

$$f_0(\alpha) > f_0(\mathbf{x})$$

Oracle for an Unconstrained Minimization Problem

$$\mathbf{g}^T(\alpha - \mathbf{x}) > 0 \Longrightarrow f_0(\alpha) > f_0(\mathbf{x})$$

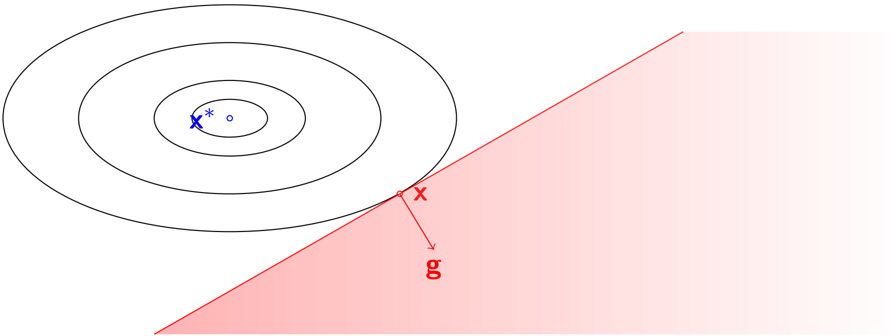
This means that all points α s.t. $\mathbf{g}^T(\alpha - \mathbf{x}) \geq 0$ are worse solutions then $\mathbf{x}$

Our oracle needs to return a cutting-plane $\mathbf{u}^T\alpha \geq v$:

$$\begin{aligned} \mathbf{g}^T(\alpha - \mathbf{x}) &\geq 0 \\ \mathbf{g}^T\alpha - \mathbf{g}^T\mathbf{x} &\geq 0 \\ \mathbf{g}^T\alpha &\geq \mathbf{g}^T\mathbf{x} \end{aligned}$$

This is a neutral cutting plane!

Subgradient as a cut criterion



Outline

Deep cut for Unconstrained Minimization

To get a deep cut we need to know a number $\bar{f}$ such that $f_0(\mathbf{x}) > \bar{f} \geq f^*$

Recall the subgradient definition: $f_0(\alpha) \geq f_0(\mathbf{x}) + \mathbf{g}^T(\alpha - \mathbf{x})$

It follows that if:

$$f_0(\mathbf{x}) + \mathbf{g}^T(\alpha - \mathbf{x}) > \bar{f}$$

then

$$f_0(\alpha) > \bar{f} \geq f^* \Longrightarrow \alpha \notin \mathcal{B}$$

which gives us the following deep cut:

$$\mathbf{g}^T(\alpha - \mathbf{x}) + f_0(\mathbf{x}) - \bar{f} \leq 0$$

Deep cut for Unconstrained Minimization

$$\mathbf{g}^T(\alpha - \mathbf{x}) + f_0(\mathbf{x}) - \bar{f} \leq 0$$

- Neutral cut plus
- Offset

How to find $\bar{f}$?

One solution: maintain the lowest value for f_0 found so far

Feasibility problem

Find a feasible $\mathbf{x} \in \mathbb{R}^n$

$$\begin{array}{rll} \text{find} & \mathbf{x} & \\ \text{subject to} & f_i(\mathbf{x}) \leq 0, & i = 1, \ldots, m \end{array}$$

For a given infeasible $\mathbf{x}$:

- get a subgradient $\mathbf{g}_j \in \partial f_j(\mathbf{x})$ for the violated constraint j: $f_j(\mathbf{x}) > 0$
- Since $f_j(\alpha) \geq f_j(\mathbf{x}) + \mathbf{g}_j^T(\alpha - \mathbf{x})$

$$f_j(\mathbf{x}) + \mathbf{g}_j^T(\alpha - \mathbf{x}) > 0 \Longrightarrow f_j(\alpha) > 0 \Longrightarrow \alpha \notin \mathcal{B}$$

- Thus every feasible $\alpha \in \mathcal{B}$ must satisfy: $f_j(\mathbf{x}) + \mathbf{g}_j^T(\alpha - \mathbf{x}) \leq 0$
- Deep cut!

Inequality constrained Problem

Now assume a general inequality constrained problem:

$$\begin{array}{rl} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \end{array}$$

Start with a point $\mathbf{x}$

- **If x** is **not feasible**, i.e. $f_j(\mathbf{x}) > 0$:
  - Perform a feasibility cut (for $\mathbf{g}_j \in \partial f_j(\mathbf{x})$):

$$f_j(\mathbf{x}) + \mathbf{g}_j^T(\alpha - \mathbf{x}) \leq 0$$

- **If x** is **feasible**:
  - Perform an objective (neutral) cut (for $\mathbf{g} \in \partial f_0(\mathbf{x})$):

$$\mathbf{g}^T(\alpha - \mathbf{x}) \leq 0$$

Inequality constrained Problem

Now assume a general inequality constrained problem:

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

Start with a point $\mathbf{x}$

- **If** $\mathbf{x}$ is **not feasible**, i.e. $f_j(\mathbf{x}) > 0$:
  - Perform a feasibility cut (for $\mathbf{g}_j \in \partial f_j(\mathbf{x})$):

$$f_j(\mathbf{x}) + \mathbf{g}_j^T(\alpha - \mathbf{x}) \leq 0$$

- **If** $\mathbf{x}$ is **feasible** and we know a number $\bar{f} : f_0(\mathbf{x}^*) \leq \bar{f} < f_0(\mathbf{x})$:
  - Perform an objective (deep) cut (for $\mathbf{g} \in \partial f_0(\mathbf{x})$):

$$\mathbf{g}^T(\alpha - \mathbf{x}) + f_0(\mathbf{x}) - \bar{f} \leq 0$$

Basic Cutting Plane Method

We start with a polyhedron $\mathcal{P}_0$ known to contain $\mathcal{B}$:

$$\mathcal{P}_0 = \{\alpha | C\alpha \succeq \mathbf{d}\}$$

We only query the oracle at points inside $\mathcal{P}_0$

For each query point we get a cutting plane $(\mathbf{u}, v)$

We get a new polyhedron by inserting the new cutting plane:

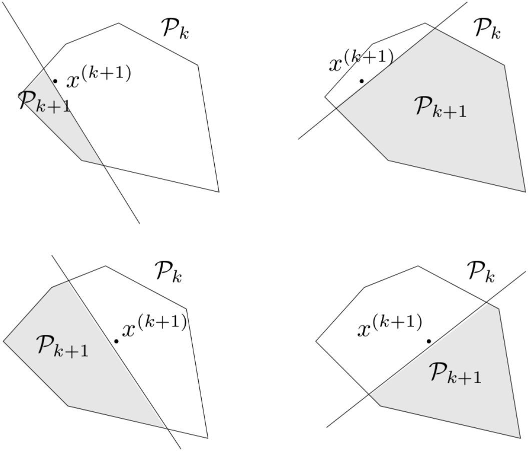
$$\mathcal{P}_{t+1} \leftarrow \mathcal{P}_t \cap \{\alpha | \mathbf{u}^T\alpha \leq v\}$$

Basic Cutting Plane Algorithm

procedure CUTTING PLANE METHOD
input: Initial Polyhedron $\mathcal{P}_0 = \{\alpha | C\alpha \succeq \mathbf{d}\}$

$t \leftarrow 0$
while not converged **do**
  Get a point $\mathbf{x}^{t+1} \in \mathcal{P}_t$
  Query the oracle at $\mathbf{x}^{t+1}$
  if $\mathbf{x}^{t+1} \in \mathcal{B}$ **then**
    return $\mathbf{x}^{t+1}$
  end if
  $\mathcal{P}_{t+1} \leftarrow \mathcal{P}_t \cap \{\alpha | \mathbf{u}_{t+1}^T \alpha \leq v_{t+1}\}$
  if $\mathcal{P}_{t+1} = \emptyset$ **then**
    Quit
  end if
  $t \leftarrow t + 1$
end while

Basic Cutting Plane Algorithm

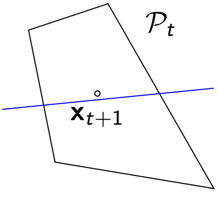


(From Stephen Boyd's Lecture Notes)

How to choose the next point

How do we choose the next $\mathbf{x}^{t+1}$?

- The size of $\mathcal{P}_{t+1}$ is a measure of our uncertainty
- We want to choose a $\mathbf{x}^{t+1}$ so that $\mathcal{P}_{t+1}$ is small as possible no matter the cut
- Strategy: choose $\mathbf{x}^{t+1}$ close to the center of $\mathcal{P}_{t+1}$



Specific Cutting Plane Methods

Specific Cutting Plane Methods differ in the choice of the query point

- Center of Gravity (CG): $\mathbf{x}^{t+1}$ is the center of gravity of $\mathcal{P}_t$
- Maximum volume ellipsoid (MVE): $\mathbf{x}^{t+1}$ is the center of the maximum volume ellipsoid contained in $\mathcal{P}_t$
- Chebyshev Center: $\mathbf{x}^{t+1}$ the Chebyshev center of $\mathcal{P}_t$
- Analytic Center: $\mathbf{x}^{t+1}$ is the analytic center of the inequalites defining $\mathcal{P}_t$

Center of gravity Method

$\mathbf{x}^{t+1}$ is the center of gravity of $\mathcal{P}_t$: $CG(\mathcal{P}_t)$

$$CG(\mathcal{P}_t) = \frac{\int_{\mathcal{P}_t} \mathbf{x} d\mathbf{x}}{\int_{\mathcal{P}_t} d\mathbf{x}}$$

Theorem: be $\mathcal{P} \subset \mathbb{R}^n$, $\mathbf{x}_{cg} = CG(\mathcal{P})$, $\mathbf{g} \neq 0$:

$$\text{vol}\left(\mathcal{P} \cap \{\mathbf{x} | \mathbf{g}^T(\mathbf{x} - \mathbf{x}_{cg}) \leq 0\}\right) \leq (1 - \frac{1}{e})\text{vol}(\mathcal{P}) \approx 0.63\text{vol}(\mathcal{P})$$

which means that, at epoch t, $\text{vol}(\mathcal{P}_t) \leq 0.63^t \text{vol}(\mathcal{P}_0)$

Maximum Volume Ellipsoid (MVE) Method

$\mathbf{x}^{t+1}$ is the center of the maximum volume ellipsoid $\mathcal{E}$ contained in $\mathcal{P}_t$

The ellipsoid can be parametrized by a positive definite matrix $E \in \mathbb{R}^{n\times n}_{++}$ and a vector $\mathbf{h} \in \mathbb{R}^n$:

$$\mathcal{E} = \{E\alpha + \mathbf{h} \mid ||\alpha||_2 \leq 1\}$$

The **Maximum Volume Ellipsoid** in a polyhedron $\{\alpha | \mathbf{c}_i^T \alpha \leq d_i, i = 1, \ldots, m\}$ can be found by solving:

$$\begin{array}{ll} \text{maximize} & \log\det E \\ \text{subject to} & ||E\mathbf{c}_i||_2 + \mathbf{c}_i^T \mathbf{h} \leq d_i, \quad i = 1, \ldots, m \end{array}$$

Maximum Volume Ellipsoid (MVE) Method

Computing the MVE is done by solving a convex optimization problem

It is affine invariant

One can show that:

$$\text{vol}(\mathcal{P}_{t+1}) \leq (1 - \frac{1}{n})\text{vol}(\mathcal{P}_t)$$

Chebyshev Center

$\mathbf{x}^{t+1}$ the center of the largest Euclidean ball in $\mathcal{P}_t$

Can be computed by linear programming:

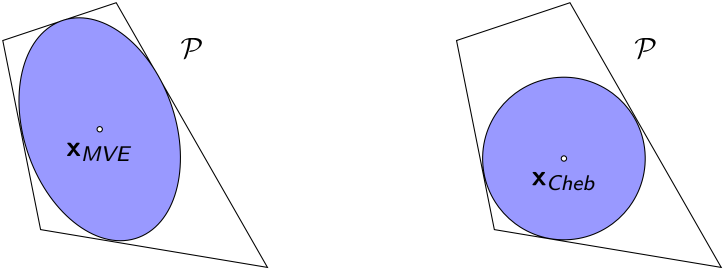
The Chebyshev center of $\{\alpha | \mathbf{c}_i^T \alpha \leq d_i, i = 1, \ldots, m\}$

Is the center of the largest ball $\{\mathbf{x}_{center} + \alpha \mid ||\alpha||_2 \leq r\}$

We can find $\mathbf{x}_{center}$ and r by solving:

$$
\begin{aligned}
\text{maximize} \quad & r \\
\text{subject to} \quad & \mathbf{c}_i^T \mathbf{x} + r||\mathbf{c}_i||_2 \leq d_i, \quad i = 1, \ldots, m
\end{aligned}
$$

MVE vs. Chebyshev Center



Analytic Center

$\mathbf{x}^{t+1}$ is the analytic center of the inequalites defining $\mathcal{P}_t$

Be $\mathcal{P}_t = \{\alpha | \mathbf{c}_i^T \alpha \leq d_i, i = 1, \ldots, q\}$:

$$\mathbf{x}^{t+1} = \arg\min_{\mathbf{x}} - \sum_{i=1}^{q} \log(d_i - \mathbf{c}_i \mathbf{x})$$

Can be solved using Newton's method