Modern Optimization Techniques

3. Equality Constrained Optimization / 3.1. Duality

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original slides by Lucas Rego Drumond (ISMLL)

Syllabus

Tue. 18.10.	(0)	0. Overview
		1. Theory
Tue. 25.10.	(1)	1. Convex Sets and Functions
		2. Unconstrained Optimization
Tue. 1.11	(2)	2.1 Gradient Descent
Tue. 8.11.	(3)	2.2 Stochastic Gradient Descent
Tue. 15.11.	(4)	(ctd.)
Tue. 22.11.	(5)	2.3 Newton's Method
Tue. 29.11.	(6)	2.4 Quasi-Newton Methods
Tue. 6.12.	(7)	2.5 Subgradient Methods
Tue. 13.12.	(8)	2.6 Coordinate Descent
		3. Equality Constrained Optimization
Tue. 20.12.	(9)	3.1 Duality
	—	— Christmas Break —
Tue. 10.1.	(10)	3.2 Methods
		4. Inequality Constrained Optimization
Tue. 17.1.	(11)	4.1 Interior Point Methods
Tue. 24.1.	(11)	4.2 Cutting Plane Method
		5. Distributed Optimization
Tue. 31.1.	(12)	5.1 Alternating Direction Method of Multipliers

Outline

1. Constrained Optimization
2. Duality
3. KKT Conditions

Outline

Constrained Optimization Problems

A **constrained optimization problem** has the form:

$$\begin{array}{rll} \text{minimize} & f_0(\mathbf{x}) & \\ \text{subject to} & f_i(\mathbf{x}) \leq 0, & i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, & i = 1, \ldots, p \end{array}$$

Where:

- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called the *objective or cost function*,
- $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are called *inequality constraints*,
- $h_1, \ldots, h_p : \mathbb{R}^n \to \mathbb{R}$ are called *equality constraints*,
- An optimal $\mathbf{x}^*$

Constrained Optimization Problems

A **convex constrained optimization problem**:

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\end{aligned}
$$

is convex iff:

- f_0, the *objective function* must be convex,
- $f_0, \ldots, f_m$ the *inequality constraint functions* must be convex,
- $h_1, \ldots, h_p$ the *equality constraint functions* must be affine, $h_j(x) = \mathbf{a}_j^T \mathbf{x} - b_j$.

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$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & \mathbf{a}_j^T\mathbf{x} = b_j, \quad j = 1, \ldots, p \end{aligned}$$

Linear Programming

A convex problem with an *affine objective* and *affine constraint* functions is called *Linear Program (LP)*.

Standard form LP:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{a}_i^T\mathbf{x} = b_i \quad i = 1, \ldots, m \\ & \mathbf{x} \succeq 0 \end{aligned}$$

Inequality form LP:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{a}_i^T\mathbf{x} \leq b_i \quad i = 1, \ldots, m \end{aligned}$$

- No analytical solution
- There are reliable algorithms available

Quadratic Programming

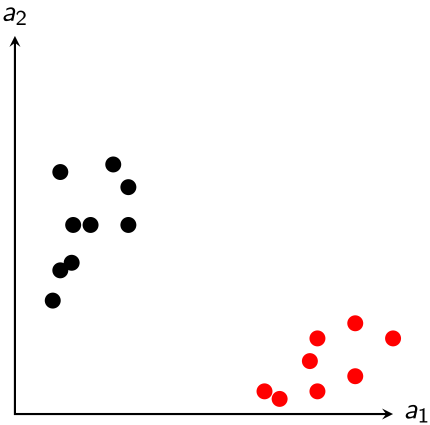
A convex problem with a *convex objective* and *affine constraint* functions is called *Quadratic Program (QP)*.

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{a}_i^T\mathbf{x} \leq b_i \quad i = 1, \ldots, m \end{array}$$

where:

- $Q \succ 0$,
- $Q = 0$, a special case, when quadratic programs include linear programs.

Maximum Margin Separating Hyperplanes



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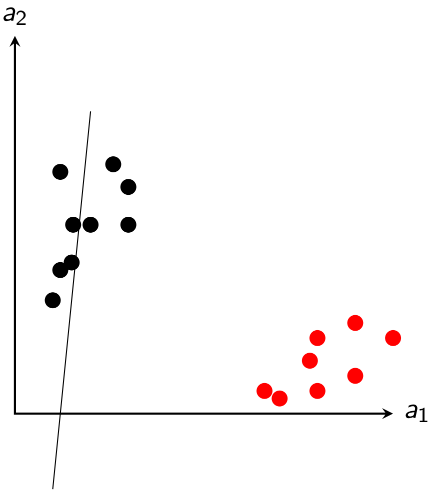

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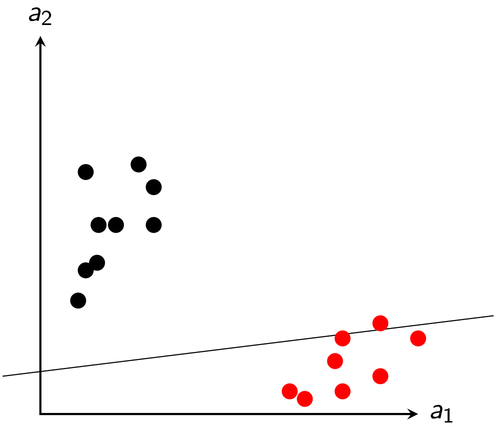

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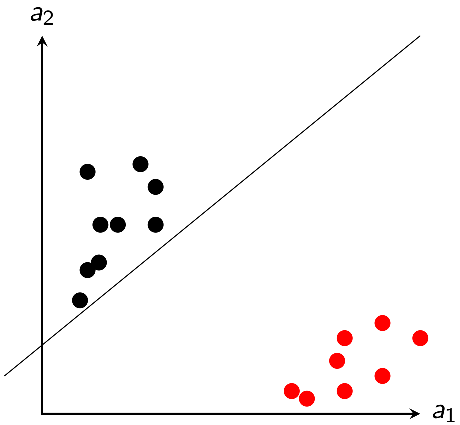

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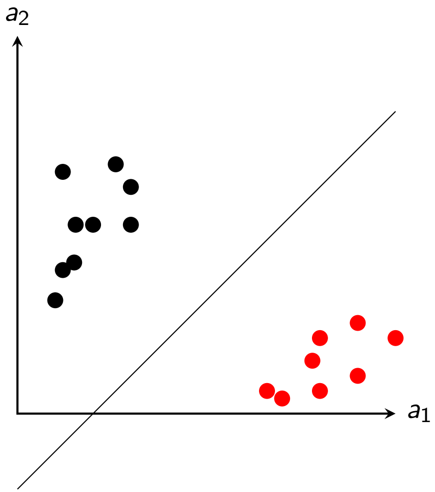

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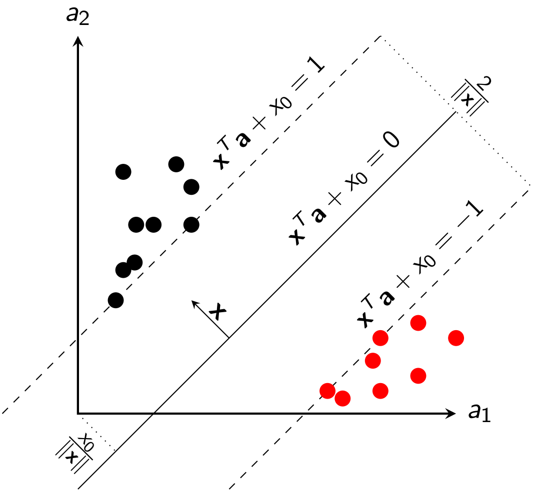

Support Vector Machines

If the instances are not completely separable, we can allow some of them to be on the wrong side of the decision boundary

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The closer the “wrong” points are to the boundary the better (modeled by slack variables ξ_i)

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}||\mathbf{x}||^2 + \gamma \sum_{i=1}^{n} \xi_i \\ \text{subject to} \quad & y_i(x_0 + \mathbf{x}^T \mathbf{a_i}) \geq 1 - \xi_i \quad i = 1, \ldots, n \\ & \xi_i \geq 0 \quad i = 1, \ldots, n \end{aligned}$$

Outline

Lagrangian

Given a constrained optimization problem in the standard form:

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We can put the objective function and the constraints in the same expression:

$$f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x})$$

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The expression above is not the same original problem. It is called the primal **Lagrangian** of the problem

Lagrangian

The **primal Lagrangian** of a constrained optimization problem is a function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

$$L(\mathbf{x}, \lambda, \nu) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{x})$$

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Dual Lagrangian

Be $\mathcal{D}$ the domain of the problem, the **dual Lagrangian** of a constrained optimization problem is a function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

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\begin{aligned}
g(\lambda, \nu) &= \inf_{\mathbf{x} \in \mathcal{D}} L(\mathbf{x}, \lambda, \nu) \\
&= \inf_{\mathbf{x} \in \mathcal{D}} \left(f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{x}) \right)
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If $\lambda \succeq 0$, then $g(\lambda, \nu) \leq f_0(\mathbf{x}^*)$

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thus, with $\lambda \succeq 0$:

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minimizing over all feasible $\mathbf{x}'$ we have $f_0(\mathbf{x}^*) \geq g(\lambda, \nu)$

Least-norm solution of linear equations

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Substituting in L to get g: $g(\nu) = -\frac{1}{4}\nu^T HH^T\nu - b^T\nu$

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- Does not always hold
- Holds for a range of convex problems
- Properties that guarantee strong duality are called constraint qualifications

Slater's Condition

If the following primal problem

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$$\exists \mathbf{x} : f_i(\mathbf{x}) < 0 \quad i = 1, \ldots, m, \qquad H\mathbf{x} = \mathbf{b}$$

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then strong duality holds for this problem

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Since $g(\lambda, \nu)$ is a lower bound on f_0:

$$f_0(\mathbf{x}) - f_0(\mathbf{x}^*) \leq f_0(\mathbf{x}) - g(\lambda, \nu)$$

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If the duality gap is zero, then $\mathbf{x}$ is primal optimal

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How close is the value of the dual lagrangian to the primal objective?

Given a primal feasible $\mathbf{x}$ and a dual feasible λ, ν, the **duality gap** is given by:

$$f_0(\mathbf{x}) - g(\lambda, \nu)$$

Since $g(\lambda, \nu)$ is a lower bound on f_0:

$$f_0(\mathbf{x}) - f_0(\mathbf{x}^*) \leq f_0(\mathbf{x}) - g(\lambda, \nu)$$

If the duality gap is zero, then $\mathbf{x}$ is primal optimal

This is a useful stopping criterion since if $f_0(\mathbf{x}) - g(\lambda, \nu) \leq \epsilon$, then we are sure that $f_0(\mathbf{x}) - f_0(\mathbf{x}^*) \leq \epsilon$

Outline

1. Constrained Optimization

2. Duality

3. KKT Conditions

Complementary Slackness

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and $\mathbf{x}^*$ minimizes $L(\mathbf{x}, \lambda^*, \nu^*)$

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If $\mathbf{x}, \lambda, \nu$ satisfy the KKT conditions, then $\mathbf{x}$ is the primal solution and (λ, ν) is the dual solution