Modern Optimization Techniques

4. Inequality Constrained Optimization / 4.3. Cutting Plane Methods

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Syllabus

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		2. Unconstrained Optimization
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1. Cutting Plane Methods: Basic Idea
2. The Oracle for Unconstrained Optimization
3. The Oracle for Constrained Optimization
4. How to Choose the Next Query Point

Outline

Inequality Constrained Minimization (ICM) Problems

Convex (but maybe **not differentiable**):

$$\begin{aligned} \arg\min_{x \in \mathbb{R}^N} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & A\mathbf{x} - a = 0 \\ & h_q(\mathbf{x}) \leq 0, \quad q = 1, \ldots, Q \end{aligned}$$

where:

- $f : \mathbb{R}^N \to \mathbb{R}$ convex, **but maybe not differentiable**
- $A \in \mathbb{R}^{P \times N}, a \in \mathbb{R}^P$: P affine equality constraints
- $h_1, \ldots, h_Q : \mathbb{R}^N \to \mathbb{R}$ convex, **but maybe not differentiable**
- A feasible optimal $\mathbf{x}^*$ exists, $p^* := f(\mathbf{x}^*)$

Let f and h be at least subdifferentiable.

I picked a number between 0 and 100.

Guess it?

I will answer one of:

- correct
- mine is lower
- mine is higher

Interval-Halving / Bisection Methods

- for example: compute $\sqrt{17}$?
- cast as optimization problem:

$$\sqrt{17} = \arg\min_{x \in \mathbb{R}} f(x), \quad f(x) := (x^2 - 17)^2$$

- can be solved by any unconstrained minimization algorithm.

```
min-bisecting(f, x+, x-, K, ε) :
    for k := 1, ..., K:
        x(k) := (x+ + x-)/2
        if |f'(x(k))| < ε:
            return x(k)
        if f'(x(k)) < 0:
            x- := x(k)
        else :
            x+ := x(k)
    return "not converged"
```

where

- $f : \mathbb{R} \to \mathbb{R}$ one-dimensional objective
- f' its derivative
- $f'(x^+) > 0$, $f'(x^-) < 0$

Interval-Halving / Bisection Methods

x^-	x^+	$x^{(k)}$	$(x^{(k)})^2 - 17$	update
0	17	8.5	55.25	x^+
0	8.5	4.25	1.0625	x^+
0	4.25	2.125	−12.484 375	x^-
2.125	4.25	3.1875	−6.839 843 75	x^-
3.1875	4.25	3.718 75	−3.170 898 437 5	x^-

Cutting Plane Methods

- Inequality constrained problems can be solved by barrier/penalty methods.
- But barrier/penalty methods assume constraints to be
  - *convex* and
  - *twice differentiable*
- What to do if h is nondifferentiable?
- **Cutting plane methods**:
  - Are able to handle nondifferentiable convex problems
  - Can also be applied to unconstrained minimization problems
  - Require the computation of a subgradient per step
  - Can be much faster than subgradient methods

Cutting Plane Methods - Basic Idea

- Let $\mathcal{B} \subseteq \mathbb{R}^N$ denote the set of all solutions $\mathbf{x}^*$ to our problem:

$$\mathcal{B} := \{\mathbf{x}^* \mid f(\mathbf{x}^*) = p^*,\ A\mathbf{x}^* - \mathbf{a} = 0,\ h(\mathbf{x}^*) \leq 0\}$$

- Assume we have an **oracle** who can "answer" $\mathbf{x} \overset{?}{\in} \mathcal{B}$
- The oracle returns a plane that separates $\mathbf{x}$ from $\mathcal{B}$
- A cutting plane method starts with an initial solution $\mathbf{x}^{(k)}$ and then:
  1. Query the oracle $\mathbf{x}^{(k)} \overset{?}{\in} \mathcal{B}$
  2. If $\mathbf{x}^{(k)} \in \mathcal{B}$ then stop and return $\mathbf{x}^{(k)}$
  3. Generate a new point $\mathbf{x}^{(k+1)}$ on the other side of the plane returned by the oracle
  4. Go back to step 1

Cutting Plane Methods - Basic Idea

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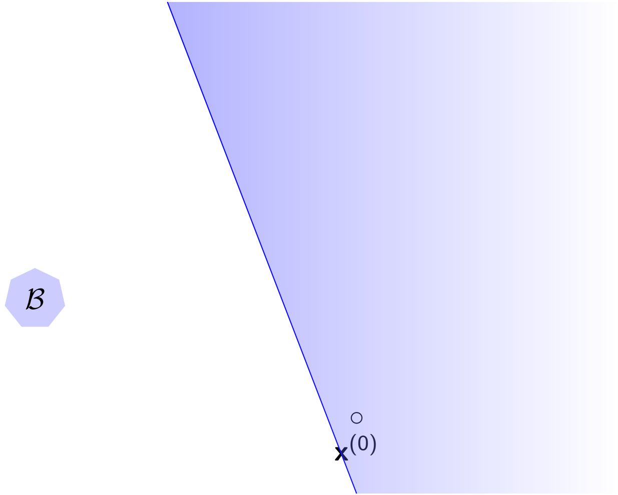
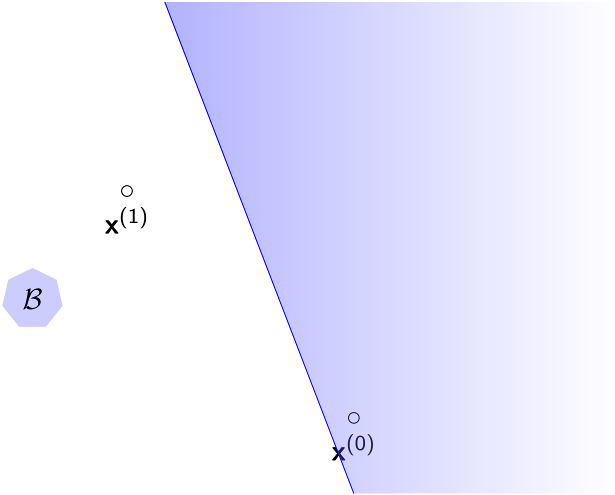
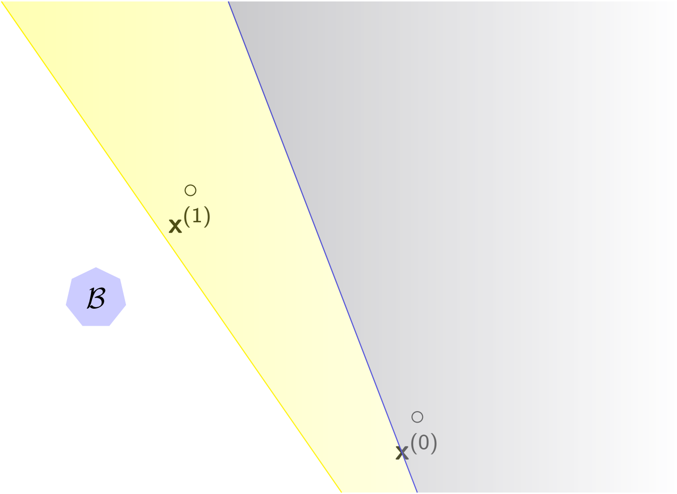
Cutting Plane Methods - Basic Idea



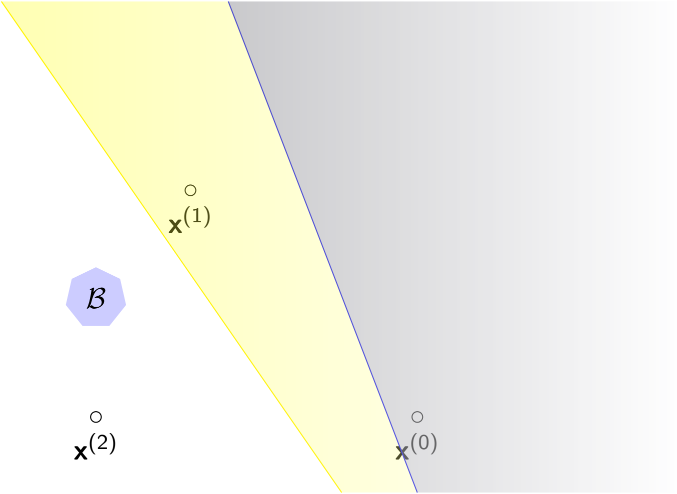
Cutting Plane Methods - Basic Idea



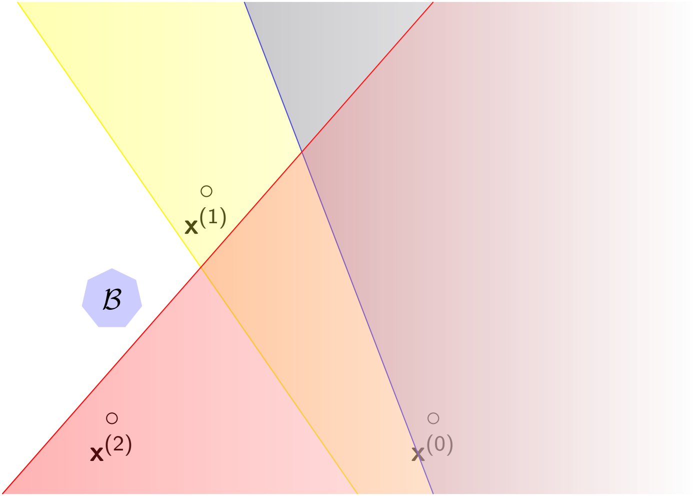
Cutting Plane Methods - Basic Idea



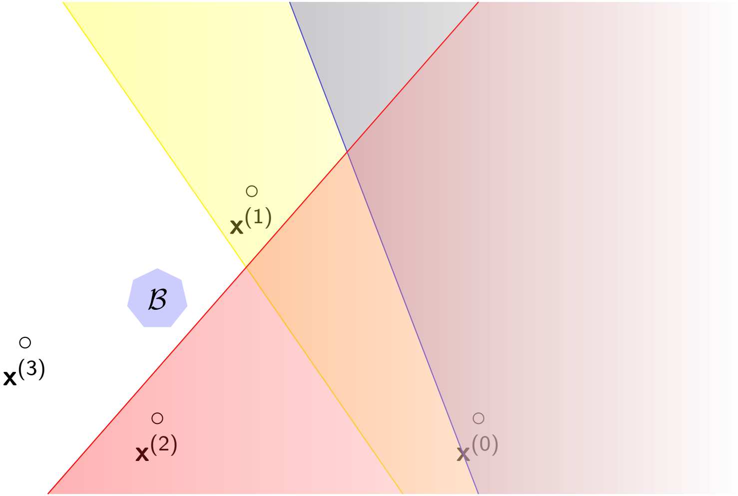
Cutting Plane Methods - Basic Idea



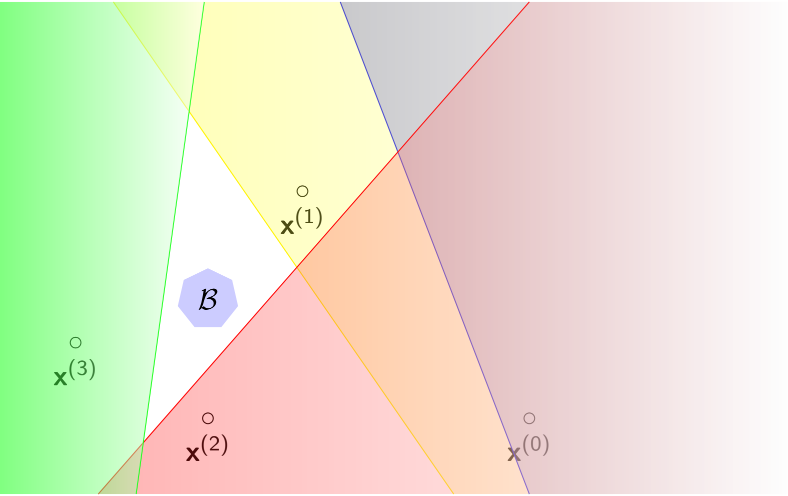
Cutting Plane Methods - Basic Idea



Cutting Plane Methods - Basic Idea



Cutting Plane Methods - Basic Idea



Outline

Cutting Plane Oracle

Goal: Determine if $\mathbf{x}^{(k)} \stackrel{?}{\in} \mathcal{B}$

- two possible outcomes of a query to the oracle:
  - a positive answer, if $\mathbf{x}^{(k)} \in \mathcal{B}$
  - a separating hyperplane $(\mathbf{u}, v)$ between $\mathbf{x}^{(k)}$ and $\mathcal{B}$, if $\mathbf{x}^{(k)} \notin \mathcal{B}$:

$$\mathbf{u}^T\mathbf{x} \leq v, \quad \text{for all } \mathbf{x} \in \mathcal{B}$$
$$\mathbf{u}^T\mathbf{x}^{(k)} \geq v$$

  with $\mathbf{u} \in \mathbb{R}^N, v \in \mathbb{R}$.

- Thus we can eliminate (cut) all points in the halfspace

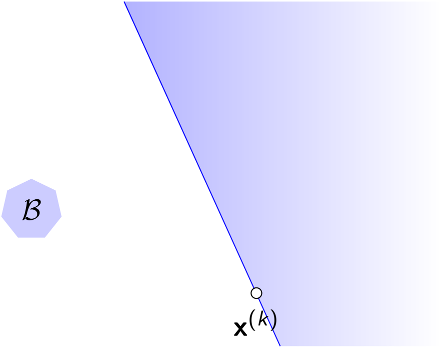
$$\text{hs}(\mathbf{u}, v) := \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{u}^T\mathbf{x} > v\}$$

  from our search.

Neutral cuts

If the query point $\mathbf{x}^{(\mathbf{k})}$ is on the boundary of the halfspace, the cut is called **neutral**:

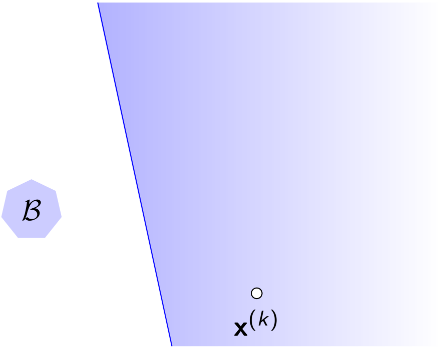
$$\mathbf{u}^T \mathbf{x}^{(k)} = v$$



Deep cuts

If the query point $\mathbf{x}^{(k)}$ is in the interior of the halfspace, the cut is called **deep**:

$$\mathbf{u}^T\mathbf{x}^{(k)} > v$$

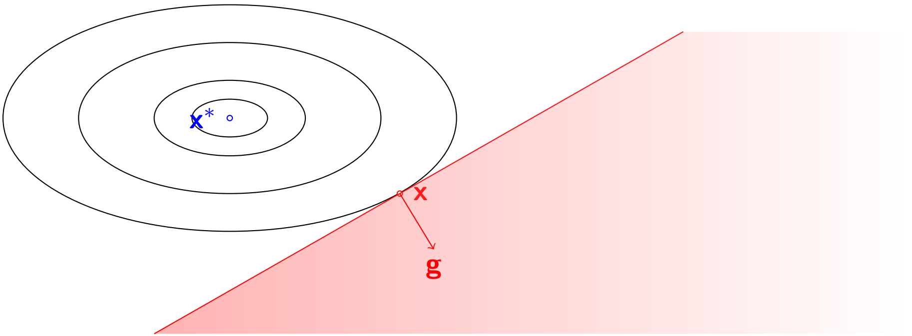


Oracle for an Unconstrained Minimization Problem

- Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be convex,
  $\mathbf{x}$ the current query point.

- The oracle can be implemented by the subdifferential $\partial f(\mathbf{x})$:
  - Remember, for any subgradient $\mathbf{g} \in \partial f(\mathbf{x})$:
  $$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{g}^T(\mathbf{y} - \mathbf{x}), \quad \forall \mathbf{y} \in \mathbb{R}^N$$
  and
  $$\mathbf{x} \in \mathcal{B} \Longleftrightarrow 0 \in \partial f(\mathbf{x})$$
  - if $0 \notin \partial f(\mathbf{x})$, then $\mathbf{x} \notin \mathcal{B}$, $\mathbf{g} \neq 0$ and
  for all $\mathbf{y}$ with $\mathbf{g}^T\mathbf{y} \geq \mathbf{g}^T\mathbf{x}$ (1):
  $$f(\mathbf{y}) \underset{\text{sg}}{\geq} f(\mathbf{x}) + \mathbf{g}^T(\mathbf{y} - \mathbf{x}) \underset{(1)}{\geq} f(\mathbf{x}) \underset{\mathbf{x} \notin \mathcal{B}}{>} f(\mathbf{x}^*) \quad \rightsquigarrow \mathbf{y} \notin \mathcal{B}$$
  - **neutral objective cut**:
  $$\text{hs}(\mathbf{g}, \mathbf{g}^T\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^N \mid \mathbf{g}^T\mathbf{y} \geq \mathbf{g}^T\mathbf{x}\} \subseteq \{\mathbf{y} \in \mathbb{R}^N \mid f(\mathbf{y}) \geq f(\mathbf{x})\}$$

Subgradient as a cut criterion



Deep Cut for Unconstrained Minimization

- Assume we know a better upper bound $\bar{f}$ for the minimal value already:

$$f(\mathbf{x}) > \bar{f} \geq f(\mathbf{x}^*)$$

- for all $\mathbf{y}$ with $\mathbf{g}^T\mathbf{y} > \mathbf{g}^T\mathbf{x} - (f(\mathbf{x}) - \bar{f})$ (1):

$$f(\mathbf{y}) \underset{\text{sg}}{\geq} f(\mathbf{x}) + \mathbf{g}^T(\mathbf{y}-\mathbf{x}) \underset{(1)}{>} \bar{f} \geq f(\mathbf{x}^*) \quad \rightsquigarrow \mathbf{y} \notin \mathcal{B}$$

- **deep objective cut**:

$$\begin{aligned} \text{hs}(\mathbf{g}, \mathbf{g}^T\mathbf{x} - (f(\mathbf{x}) - \bar{f})) &= \{\mathbf{y} \in \mathbb{R}^N \mid \mathbf{g}^T\mathbf{y} > \mathbf{g}^T\mathbf{x} + \bar{f} - f(\mathbf{x})\} \\ &\subseteq \{\mathbf{y} \in \mathbb{R}^N \mid f(\mathbf{y}) > \bar{f}\} \end{aligned}$$

- To get $\bar{f}$, maintain the lowest value for f found so far:

$$\bar{f}^{(k)} := \min_{k'=1,\ldots,k-1} f(\mathbf{x}^{(k')})$$

Outline

Feasility Problems

Find a feasible $\mathbf{x} \in \mathbb{R}^N$

$$\begin{aligned} \text{find} \quad & \mathbf{x} \\ \text{subject to} \quad & h(\mathbf{x}) \leq 0 \end{aligned}$$

i.e., $\mathbf{x} \in \mathcal{B} := \{\mathbf{x} \in \mathbb{R}^N \mid h(\mathbf{x}) \leq 0\}$.

For a given infeasible $\mathbf{x}$:

- let constraint q be violated: $h_q(\mathbf{x}) > 0$
  and $\mathbf{g}_q \in \partial h_q(\mathbf{x})$ be one of its subgradients.
- for $\mathbf{y}$ with

$$\mathbf{g}_q{}^T \mathbf{y} > \mathbf{g}_q^T \mathbf{x} - h_q(\mathbf{x}) \tag{1}$$

$$\rightsquigarrow \quad h_q(\mathbf{y}) \underset{\text{sg}}{\geq} h_q(\mathbf{x}) + \mathbf{g}_q^T(\mathbf{y} - \mathbf{x}) \underset{(1)}{>} 0 \quad \rightsquigarrow \mathbf{y} \notin \mathcal{B}$$

- deep **feasibility cut**:

$$\text{hs}(\mathbf{g}_q, \mathbf{g}_q^T \mathbf{x} - h_q(\mathbf{x})) = \{\mathbf{y} \mid \mathbf{g}_q^T \mathbf{y} > \mathbf{g}_q^T \mathbf{x} - h_q(\mathbf{x})\} \subseteq \{\mathbf{y} \mid h_q(\mathbf{x}) > 0\}$$

Inequality Constrained Problems

- general inequality constrained problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & h(\mathbf{x}) \leq 0 \end{aligned}$$

- start with a point $\mathbf{x}$:
  - **If** $\mathbf{x}$ is not feasible, i.e. $h_q(\mathbf{x}) > 0$:
    - Perform a deep feasibility cut (for $\mathbf{g}_q \in \partial h_q(\mathbf{x})$):

$$\text{hs}(\mathbf{g}_q, \mathbf{g}_q^T \mathbf{x} - h_q(\mathbf{x})) = \{\mathbf{y} \mid \mathbf{g}_q^T \mathbf{y} > \mathbf{g}_q^T \mathbf{x} - h_q(\mathbf{x})\} \subseteq \{\mathbf{y} \mid h_q(\mathbf{x}) > 0\}$$

  - **If** $\mathbf{x}$ is feasible ($\mathbf{g} \in \partial f(\mathbf{x})$):
    - if we know a better upper bound $\bar{f}$ for the optimal value, i.e., $f(\mathbf{x}) > \bar{f} \geq f(\mathbf{x}^*)$: perform a deep objective cut:

$$\begin{aligned} \text{hs}(\mathbf{g}, \mathbf{g}^T \mathbf{x} - (f(\mathbf{x}) - \bar{f})) = \{\mathbf{y} \in \mathbb{R}^N \mid \mathbf{g}^T \mathbf{y} > \mathbf{g}^T \mathbf{x} + \bar{f} - f(\mathbf{x})\} \\ \subseteq \{\mathbf{y} \in \mathbb{R}^N \mid f(\mathbf{y}) > \bar{f}\} \end{aligned}$$

    - otherwise: perform a neutral objective cut:

$$\text{hs}(\mathbf{g}, \mathbf{g}^T \mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^N \mid \mathbf{g}^T \mathbf{y} \geq \mathbf{g}^T \mathbf{x}\} \subseteq \{\mathbf{y} \in \mathbb{R}^N \mid f(\mathbf{y}) \geq f(\mathbf{x})\}$$

General Cutting Plane Method

- We start with a polyhedron $\mathcal{P}^{(0)}$ known to contain $\mathcal{B}$:

$$\mathcal{P}^{(0)} = \{\mathbf{x} \mid C^{(0)}\mathbf{x} \leq \mathbf{d}^{(0)}\}$$

- We only query the oracle at points inside $\mathcal{P}^{(0)}$

- For each query point we get a cutting plane $(\mathbf{u}, v)$

- We get a new polyhedron by inserting the new cutting plane:

$$\mathcal{P}^{(k+1)} := \mathcal{P}^{(k)} \cap \{\mathbf{x} \mid \mathbf{u}^T\mathbf{x} \leq v\} = \{\mathbf{x} \mid C^{(k+1)}\mathbf{x} \leq \mathbf{d}^{(k+1)}\}$$

with $C^{(k+1)} := \begin{bmatrix} C^{(k)} \\ u^T \end{bmatrix}, \quad \mathbf{d}^{(k+1)} := \begin{bmatrix} d^{(k)} \\ v \end{bmatrix}$

min-cuttingplane$(f, \partial f, h, \partial h, C^{(0)}, d^{(0)}, \epsilon, K^{\text{back}}, K)$:
$f_{\min} := \infty$
for $k := 1, \ldots, K$:
$x^{(k)} :=$ compute-next-query$(C^{(k-1)}, d^{(k-1)})$
if $k > K^{\text{back}}$ and $||x^{(k)} - x^{(k-K^{\text{back}})}|| < \epsilon$:
return $x^{(k)}$
if not $h(x^{(k)}) \leq 0$:
choose q with $h_q(x^{(k)}) > 0$
choose $g \in \partial h_q(x^{(k)})$
$u := g, \quad v := g^T x^{(k)} - h_q(x^{(k)})$
else :
choose $g \in \partial f(x^{(k)})$
if $f(x^{(k)}) \geq f_{\min}$:
$u := g, \quad v := g^T x^{(k)} - (f(x^{(k)}) - f_{\min})$
else :
$f_{\min} := f(x^{(k)})$
$u := g, \quad v := g^T x^{(k)}$
$C^{(k)} := \begin{bmatrix} C^{(k-1)} \\ u^T \end{bmatrix}, \quad d^{(k)} := \begin{bmatrix} d^{(k-1)} \\ v \end{bmatrix}$
return "not converged"

General Cutting Plane Method / Arguments

Algorithm arguments:

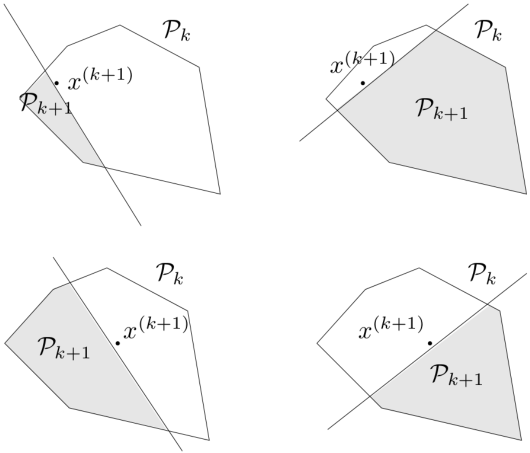
- $f : \mathbb{R}^N \rightarrow \mathbb{R}, \partial f$ objective function and its subgradient
- $h : \mathbb{R}^N \rightarrow \mathbb{R}^Q, \partial h$ inequality constraints, $h(x) \leq 0$, and its subgradient
- $C^{(0)} \in \mathbb{R}^{N \times R}, d^{(0)} \in \mathbb{R}^R$ starting polyhedron (containing the solution x^*)

Discussion:

- usually a better convergence criterion is needed.
- additional affine equality constraints can be passed through to compute-next-query.

Outline

How to Choose the Next Query Point

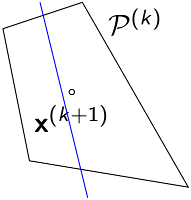


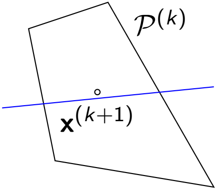
(From Stephen Boyd's Lecture Notes)

How to choose the next point

How do we choose the next $\mathbf{x}^{(k+1)}$?

- The size of $\mathcal{P}^{(k+1)}$ is a measure of our uncertainty
- We want to choose a $\mathbf{x}^{(k+1)}$ so that $\mathcal{P}^{(k+1)}$ is small as possible no matter the cut
- Strategy: choose $\mathbf{x}^{(k+1)}$ close to the center of $\mathcal{P}^{(k)}$





Specific Cutting Plane Methods

Specific cutting plane methods differ in the choice of the next query point $\mathbf{x}^{(k)}$:

- center of gravity (CG) of $\mathcal{P}^{(k)}$.
- center of the maximum volume ellipsoid (MVE) contained in $\mathcal{P}^{(k)}$.
- center of the maximum volume sphere contained in $\mathcal{P}^{(k)}$ (Chebyshev center).
- analytic center of the inequalites defining $\mathcal{P}^{(k)}$.

Methods differ in

- guarantees they provide for the decrease in volume of $\mathcal{P}^{(k+1)}$.
- how difficult they are to compute.

Center of Gravity Method

- $\mathbf{x}^{(k+1)}$ is the center of gravity of $\mathcal{P}^{(k)}$: $CG(\mathcal{P}^{(k)})$

$$CG(\mathcal{P}^{(k)}) = \frac{\int_{\mathcal{P}^{(k)}} \mathbf{x}\, d\mathbf{x}}{\int_{\mathcal{P}^{(k)}} 1\, d\mathbf{x}}$$

- **Theorem**: be $\mathcal{P} \subset \mathbb{R}^N$, $\mathbf{x}_{cg} = CG(\mathcal{P})$, $\mathbf{g} \neq 0$:

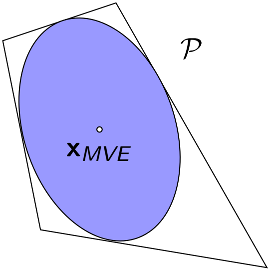
$$\text{vol}\left(\mathcal{P} \cap \{\mathbf{x} \mid \mathbf{g}^T(\mathbf{x} - \mathbf{x}_{cg}) \leq 0\}\right) \leq (1 - \frac{1}{e})\,\text{vol}(\mathcal{P}) \approx 0.63\,\text{vol}(\mathcal{P})$$

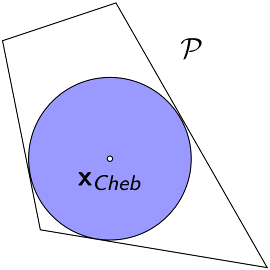
  thus at step k:

$$\text{vol}(\mathcal{P}^{(k)}) \leq 0.63^k\,\text{vol}(\mathcal{P}^{(0)})$$

- In general, it is difficult to compute the center of gravity.

Maximum Volume Ellipsoid (MVE) vs. Maximum Volume Sphere (Chebyshev Center)





Maximum Volume Ellipsoid (MVE) Method

$\mathbf{x}^{(k+1)}$ is the center of the maximum volume ellipsoid $\mathcal{E}$ contained in $\mathcal{P}^{(k)}$. Such an ellipsoid can be parametrized by

- a positive definite matrix $E \in \mathbb{R}^{N \times N}_{++}$ and
- a vector $\mathbf{h} \in \mathbb{R}^N$:

$$\mathcal{E}(E, \mathbf{h}) := \{E\alpha + \mathbf{h} \mid \alpha \in \mathbb{R}^N, ||\alpha||_2 \leq 1\}$$

The **Maximum Volume Ellipsoid** in a polyhedron

$$\mathcal{P}^{(k)} = \{\mathbf{x} \mid \mathbf{c}_r^T \mathbf{x} \leq d_r, r = 1, \ldots, R\}$$

can be found by solving:

$$\begin{array}{ll} \text{maximize} & \log\det E \\ \text{subject to} & ||E\mathbf{c}_r||_2 + \mathbf{c}_r^T \mathbf{h} \leq d_r, \quad r = 1, \ldots, R \end{array}$$

Maximum Volume Ellipsoid (MVE) Method

- The MVE is affine invariant.
- The MVE can be computed by solving a convex optimization problem.
- Volumes decrease as follows:

$$\text{vol}(\mathcal{P}^{(k+1)}) \leq (1 - \frac{1}{N})\,\text{vol}(\mathcal{P}^{(k)})$$

Chebyshev Center

- $\mathbf{x}^{(k+1)}$ the center of the largest Euclidean ball

$$\mathcal{S}(\rho, \mathbf{x}_{\text{center}}) := \{\mathbf{x}_{center} + \mathbf{x} \mid ||\mathbf{x}||_2 \leq \rho\}$$

contained in

$$\mathcal{P}^{(k)} = \{\mathbf{x} \mid \mathbf{c}_r^T \mathbf{x} \leq d_r, r = 1, \ldots, R\}$$

- Can be computed by linear programming:

$$\begin{array}{rl} \text{maximize} & \rho \\ \text{subject to} & \mathbf{c}_r^T \mathbf{x}_{\text{center}} + \rho||\mathbf{c}_r||_2 \leq d_r, \quad r = 1, \ldots, R \\ & \rho \geq 0 \\ \text{over} & \mathbf{x}_{\text{center}} \in \mathbb{R}^N, \quad \rho \in \mathbb{R} \end{array}$$

Analytic Center

- $\mathbf{x}^{(k+1)}$ is the analytic center of the inequalites defining $\mathcal{P}^{(k)}$:

$$\mathcal{P}^{(k)} = \{\mathbf{x} \mid \mathbf{c}_r^T \mathbf{x} \leq d_r, r = 1, \ldots, R\}$$

$$\mathbf{x}^{(k+1)} = \arg\min_{\mathbf{x}} - \sum_{r=1}^{R} \log(d_r - \mathbf{c}_r^T \mathbf{x})$$

- can be solved using any unconstrained method.
  - e.g., Newton's method
  - but requires a feasible starting point, i.e., $d_r - \mathbf{c}_r^T \mathbf{x} > 0 \; \forall r$. (e.g., computed via a phase 1 method).

Summary (1/2)

- **Cutting plane methods** can solve
  inequality constrained, **convex**, **subdifferentiable**
  optimization problems (that do not have to be differentiable).

- The solution is **boxed in a polyhedron**, i.e., a set of hyperplanes.

- In each step, one hyperplane is added to the boundary of the polyhedron (**oracle**).
  - if a constraint is violated: a hyperplane cutting points with higher constraint violation value (**deep feasibility cut**).
  - if the current point has larger objective function value than earlier points: a hyperplane cutting points with higher objective function value than the lowest one observed so far (**deep objective cut**).
  - otherwise: a hyperplane cutting points with higher objective function value than the current point (**neutral objective cut**).

Summary (2/2)

- To **choose the next query point**, methods **finding a center point** are interesting that
  - guarantee to cut large parts of the current polyhedron (whatever the outcome of the oracle)
  - are fast/easy to compute.
- Different methods exist to choose the next query point:
  - **center of gravity**
  - **maximum volume ellipsoid**
  - **maximum volume sphere** (Chebyshev center)
  - **analtyic center**

Further Readings

- Cutting plane methods are not covered by Boyd and Vandenberghe [2004].

- Cutting plane methods:
  - [Luenberger and Ye, 2008, ch. 14.8]

- Cutting plane methods are not covered by Griva et al. [2009] and Nocedal and Wright [2006] either.

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