Modern Optimization Techniques

2. Unconstrained Optimization / 2.6. Coordinate Descent

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Syllabus

Mon. 28.10.	(0)	0. Overview
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		2. Unconstrained Optimization
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		3. Equality Constrained Optimization
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Outline

1. Idea and Optimality
2. Coordinate Descent Algorithm
3. Convergence

Outline

Coordinate Descent

- Gradient Descent and Stochastic Gradient Descent:
  - update **all** variables simultaneously.
  - use the gradient to do so.
    (first order methods)
- **Coordinate Descent**:
  - update **one** variable at a time.
  - use an **analytic solver** to do so
    (**derivative-free method**)
    - if not possible: use one-dimensional gradient steps
      (first order method; often slow)

Coordinate Descent

We start with an initial guess $\mathbf{x}^{(0)}$

For $k = 1, 2, 3, \ldots$:

Coordinate Descent

We start with an initial guess $\mathbf{x}^{(0)}$

For $k = 1, 2, 3, \ldots$:

$$\mathbf{x}_1^{(k)} \leftarrow \arg\min_{x_1} f(x_1, \mathbf{x}_2^{(k-1)}, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_n^{(k-1)})$$

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For $k = 1, 2, 3, \ldots$:

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$$\mathbf{x}_2^{(k)} \leftarrow \arg\min_{x_2} f(\mathbf{x}_1^{(k)}, x_2, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_n^{(k-1)})$$

Coordinate Descent

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$$\mathbf{x}_3^{(k)} \leftarrow \arg\min_{x_3} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, x_3, \ldots, \mathbf{x}_n^{(k-1)})$$

Coordinate Descent

We start with an initial guess $\mathbf{x}^{(0)}$

For $k = 1, 2, 3, \ldots$:

$$\mathbf{x}_1^{(k)} \leftarrow \arg\min_{x_1} f(x_1, \mathbf{x}_2^{(k-1)}, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_n^{(k-1)})$$

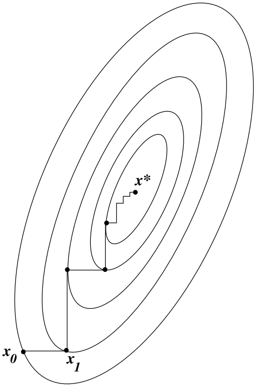
$$\mathbf{x}_2^{(k)} \leftarrow \arg\min_{x_2} f(\mathbf{x}_1^{(k)}, x_2, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_n^{(k-1)})$$

$$\mathbf{x}_3^{(k)} \leftarrow \arg\min_{x_3} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, x_3, \ldots, \mathbf{x}_n^{(k-1)})$$

$$\vdots$$

$$\mathbf{x}_n^{(k)} \leftarrow \arg\min_{x_n} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \mathbf{x}_3^{(k)}, \ldots, x_n)$$

Coordinate Descent Algorithm



[Nocedal and Wright, 2006, p.249]

Does Coordinate-wise Minimization Lead to Global Minima?

Question: If a point $\mathbf{x}$ is minimal along each axis, is it a global minimum?

$$f(\mathbf{x} + t\,\mathbf{e}^{(\mathbf{n})}) \geq f(\mathbf{x}) \quad \forall t \in \mathbb{R}, \forall n \in \{1, \ldots, N\}$$
$$\stackrel{?}{\Longrightarrow} f(\mathbf{x}) = \min_{\mathbf{y}} f(\mathbf{y})$$

where:

- $\mathbf{e}^{(\mathbf{n})} \in \mathbb{R}^N$ is the n-th unit vector with $\mathbf{e}_n^{(n)} := 1$ and $\mathbf{e}_m^{(n)} := 0$ for $m \neq n$.

Does Coordinate-wise Minimization Lead to Global Minima?

If f is **convex** and **differentiable**: yes.

Proof: $g^{(i)}(\mu) := f(\mathbf{x} + \mu\,\mathbf{e^{(i)}})$ are convex and differentiable.
$\mu = 0$ is their minimum, thus

$$0 = \frac{\partial g^{(i)}}{\partial \mu}(\mu) = \frac{\partial f}{\partial \mathbf{x}_i}(\mathbf{x})$$

And then

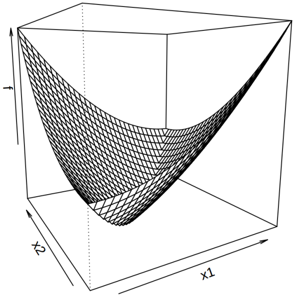
$$\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_1}, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_2}, \ldots, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_n} \right)^T = \mathbf{0}$$

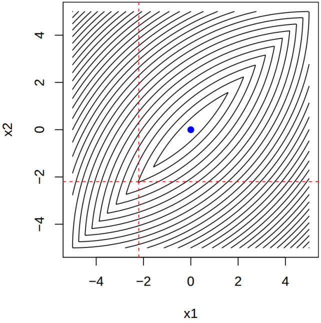
and thus x is the global optimum.

Does Coordinate-wise Minimization Lead to Global Minima?

If f is **convex**, but **not differentiable**: in general, no.

Counter example:





https://www.cs.cmu.edu/~ggordon/10725-F12/slides/25-coord-desc.pdf

Does Coordinate-wise Minimization Lead to Global Minima?

If f is a sum of a differentiable convex and a separable convex function: yes.

$$f(\mathbf{x}) = g(\mathbf{x}) + \sum_{n=1}^{N} h_n(\mathbf{x}_n)$$

with

i. g is differentiable and convex

ii. all h_n are convex
(but each h_n depends only on a single $\mathbf{x}_n$)

Does Coordinate-wise Minimization Lead to Global Minima? / Proof

Proof: For any $\mathbf{y} \in \operatorname{dom} f$:

$$f(\mathbf{y}) - f(\mathbf{x}) \underset{i.}{\geq} \nabla g(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \sum_{n=1}^{N}(h_n(\mathbf{y}_n) - h_n(\mathbf{x}_n))$$

$$= \sum_{n=1}^{N} \underbrace{\nabla_n g(\mathbf{x})(\mathbf{y}_n - \mathbf{x}_n) + h_n(\mathbf{y}_n) - h_n(\mathbf{x}_n)}_{\geq 0} \geq 0$$

where $\geq$ is argued as follows: as x is minimal along axis n,

$\leadsto$ $f(X_n; X_{-n})$ is convex and has minimum at $X_n = x_n$

$\leadsto$ 0 is a subgradient of $f(X_n; X_{-n})$ at x_n :

$$0 \in \partial f(X_n; X_{-n}) = \nabla g(x) + \partial h_n(x_n)$$

$\leadsto$ $\exists s \in \partial h_n(x_n) : \nabla g(x) + s = 0$

$$\nabla_n g(x)(y_n - x_n) + h_n(y_n) - h_n(x_n)$$

$$\underset{ii.\text{ sub grad}}{\geq} \nabla g(x)(y_n - x_n) + s(y_n - x_n) = 0$$

Outline

One-dimensional Subproblems

Let $f : X \rightarrow \mathbb{R}, X \subseteq \mathbb{R}^N$ be a function, $x \in \operatorname{dom} f$ a point.
***n*-th one-dimensional subproblem at x:**

$$
\begin{aligned}
g_n^{(x)} &: T_n^{(x)} \rightarrow \mathbb{R} \\
g_n^{(x)}(t) &:= f(x + t\, e^{(n)}) \\
T_n^{(x)} &:= \{t \in \mathbb{R} \mid x + t\, e^{(n)} \in \operatorname{dom} f\}
\end{aligned}
$$

***n*-th one-dimensional subproblem solver at x:**

$$
h_n(x) := \arg\min_{t \in T_n^{(x)}} g_n^{(x)}(t)
$$

One-dimensional Subproblems / Example

Solving a linear system of equations / least squares / linear regression:

$$f(x) := x^T A x - b^T x, \quad A \in \mathbb{R}^{N\times N} \text{ sym. pos.def.}, b \in \mathbb{R}^N$$

$$g_n^{(x)}(t) = f(x_n = t; x_{-n}), \quad T_n^{(x)} = \mathbb{R}$$

$$f(x_n; x_{-n}) = A_{n,n} x_n^2 + (2(x_{-n}^T A_{-n,.})_n - b_n) x_n + x_{-n}^T A_{-n,-n} x_{-n} - b_{-n}^T x_{-n}$$

$$= A_{n,n} x_n^2 + (2A_{n,-n} x_{-n} - b_n) x_n + x_{-n}^T A_{-n,-n} x_{-n} - b_{-n}^T x_{-n}$$

analytic minimum:

$$f'(x_n; x_{-n}) = 2A_{n,n} x_n + 2A_{n,-n} x_{-n} - b_n \stackrel{!}{=} 0$$

$$\rightsquigarrow \quad x_n = \frac{b_n - 2A_{n,-n} x_{-n}}{2A_{n,n}}$$

i.e., $h_n(x) := \dfrac{b_n - 2A_{n,-n} x_{-n}}{2A_{n,n}}$

Coordinate Descent / Random Coordinate

min−cd($f, h, x^{(0)}, K, \epsilon$):
  for $k := 1, \ldots, K$:
    draw $n^{(k)} \sim \text{unif}(\{1, \ldots, N\})$
    $x_m^{(k)} := x_m^{(k-1)}$ for $m \in \{1, \ldots, N\}, m \neq n^{(k)}$
    $x_{n^{(k)}}^{(k)} := h_{n^{(k)}}(x^{(k-1)})$
    if $k \geq N$ and $||x^{(k)} - x^{(k-N)}|| \leq \epsilon$
      return $x^{(k)}$
  return "not converged"

where

- h solvers for one-dimensional subproblems $g_n^{(x)}$.
- ϵ convergence threshold step in an epoche

Coordinate Descent / Cyclic Epochs

$\text{min-cd}(f, h, x^{(0)}, K, \epsilon)$:
  for $k := 1, \ldots, K$:
    $x_1^{(k)} := h_1(x_1^{(k-1)}, \ldots, x_N^{(k-1)})$
    $x_2^{(k)} := h_2(x_1^{(k)}, x_2^{(k-1)}, \ldots, x_N^{(k-1)})$
    $x_3^{(k)} := h_3(x_1^{(k)}, x_2^{(k)}, x_3^{(k-1)}, \ldots, x_N^{(k-1)})$
    $\vdots$
    $x_N^{(k)} := h_N(x_1^{(k)}, \ldots, x_{N-1}^{(k)}, x_N^{(k-1)})$
    if $||x^{(k)} - x^{(k-1)}|| \leq \epsilon$
      return $x^{(k)}$
  return "not converged"

where

- h solvers for one-dimensional subproblems $g_n^{(x)}$.
- ϵ convergence threshold step in an epoche

Coordinate Descent / Cyclic Epochs

min−cd$(f, h, x^{(0)}, K, \epsilon)$:
    $x^{(0,N)} := x^{(0)}$
    for $k := 1, \ldots, K$:
        $x^{(k,0)} := x^{(k-1,N)}$
        for $n := 1, \ldots, N$:
            $x_m^{(k,n)} := x_m^{(k,n-1)}$ for $m \in \{1, \ldots, N\}, m \neq n$
            $x_n^{(k,n)} := h_n(x^{(k,n-1)})$
        if $||x^{(k,N)} - x^{(k-1,N)}|| \leq \epsilon$
            return $x^{(k,N)}$
    return "not converged"

where

- h solvers for one-dimensional subproblems $g_n^{(x)}$.
- ϵ convergence threshold step in an epoche

Coordinate Descent / 1-dim Gradient Steps

$\text{min}-\text{cd}(f, \nabla f, x^{(0)}, K, \epsilon)$:
  for $k := 1, \ldots, K$:
    draw $n^{(k)} \sim \text{unif}(\{1, \ldots, N\})$
    $x^{(k)} := x^{(k-1)} - \mu^{(k)} (\nabla f(x^{(k-1)}))_{n^{(k)}} e^{(n^{(k)})}$
    if $k \geq N$ and $||x^{(k)} - x^{(k-N)}|| \leq \epsilon$
      return $x^{(k)}$
  return "not converged"

where

- ∇f gradients of objective function.
- $\mu \in (\mathbb{R}^+)^*$ step length schedule (or controller)
- ϵ convergence threshold step in an epoche

Coordinate Descent - Considerations

- The order in which we cycle through the coordinates is arbitrary.
  - e.g., cyclic
  - better should be randomized.
- We may update blocks of coordinates at a time instead of only one (**block coordinate descent**)
- No need to adjust a step-size!
  - if we have exact solvers h for the 1-dim. subproblems.
- Does not work in general with non-differentiable functions

Outline

Coordinate Lipschitz Constant

- standard Lipschitz constant L:

$$||\nabla f(x) - \nabla f(y)|| \leq L||x-y|| \qquad \forall x,y \in \text{dom } f$$

or equivalently

$$||\nabla f(x+d) - \nabla f(x)|| \leq L||d|| \qquad \forall x \in \text{dom } f, d \in \mathbb{R}^N : x+d \in \text{dom } f$$

- **component Lipschitz constant** L_n: $(n \in \{1,\ldots,N\})$

$$||\nabla f(x+t\, e^{(n)}) - \nabla f(x)|| \leq |t|\, L_n \qquad \forall x \in \text{dom } f, t \in \mathbb{R} : x + t\, e^{(n)} \in \text{dom } f$$

- **coordinate Lipschitz constant** $L_{\max}$:

$$L_{\max} := \max_{n \in \{1,\ldots,N\}} L_n$$

Note: In the following $||\cdot||$ denotes the L2-norm.

Lipschitz Continuous Functions / Bounded Derivative

Lemma

A differentiable function $f : X \to \mathbb{R}, X \subseteq \mathbb{R}$ is Lipschitz-continuous iff its derivative is bounded:

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in \text{dom}\, f \quad \iff \quad |f'(x)| \leq L \quad \forall x \in \text{dom}\, f$$

Proof: "$\Rightarrow$":

$$\left|\frac{f(x) - f(x+t)}{|t|}\right| = \frac{|f(x) - f(x+t)|}{|t|} \leq \frac{|t|L}{|t|} \leq L \quad |t \to 0$$

$$|f'(x)| \leq L$$

"$\Leftarrow$":

$$f(x) \underset{Taylor}{=} f(y) + f'(\xi)(x-y) \quad \textit{for a } \xi \in [x, y]$$

$$\leq f(y) + L(x-y)$$

$$|f(x) - f(y)| \leq L|x-y|$$

Theorem (convergence of coordinate descent)

If

i. $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^N$ *is convex and differentiable,*

ii. ∇f *is uniformly Lipschitz-continuous:*
$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \quad L \in \mathbb{R}_0^+$

iii. *the sublevel set of* $x^{(0)}$ *is bounded:*
$\max_{x^*: f(x^*)=\min_x f(x)} \max_{x \in S_f(x^{(0)})} ||x - x^*|| \leq R, \quad R \in \mathbb{R}_0^+$

iv. *constant steplength* $\mu^{(k)} := 1/L_{\max}$ *is used,*

then coordinate descent converges and

$$\mathbb{E}(f(x^{(k)})) - f(x^*) \leq \frac{2NL_{\max}R^2}{k}$$

Convergence / Proof (1/3)

$$
\begin{aligned}
f(x^{\text{next}}) &= f(x - \mu(\nabla f(x))_n e^{(n)}) \\
&\underset{\text{Taylor}}{=} f(x) - \nabla f(x)^T \mu(\nabla f(x))_n e^{(n)} \\
&\quad + \frac{1}{2}(\mu(\nabla f(x))_n e^{(n)})^T \underbrace{\nabla^2 f(x - \xi\mu(\nabla f(x))_n e^{(n)})}_{\underset{\text{ii. bound.deriv.}}{\leq} L_n} \mu(\nabla f(x))_n e^{(n)} \\
&\leq f(x) - \nabla f(x))_n^2 \mu + \frac{1}{2}\mu^2(\nabla f(x))_n^2 L_n \\
&= f(x) - (\nabla f(x))_n^2 \mu(1 - \frac{1\mu L_n}{2}) \\
&\leq f(x) - (\nabla f(x))_n^2 \mu(1 - \frac{1\mu L_{\max}}{2}) \\
&\underset{iv.}{=} f(x) - \frac{(\nabla f(x))_n^2}{2L_{\max}}
\end{aligned}
$$

Convergence / Proof (2/3)

$$f(x^{\text{next}}) \leq f(x) - \frac{(\nabla f(x))_n^2}{2L_{\max}} \quad |\mathbb{E}_n(\ldots)$$

$$\mathbb{E}_n(f(x^{\text{next}})) \leq \mathbb{E}_n(f(x) - \frac{(\nabla f(x))_n^2}{2L_{\max}})$$

$$= f(x) - \frac{1}{N}\sum_{n=1}^{N} \frac{(\nabla f(x))_n^2}{2L_{\max}}) = f(x) - \frac{1}{2NL_{\max}}||\nabla f(x)||^2 \quad (1)$$

$$\mathbb{E}_{n^{(0:K)}} := \mathbb{E}_{n^{(0)},n^{(1)},\ldots,n^K} \quad \text{expectation over } \textbf{all } n^{(k)}$$

$$\phi^{(k)} := \mathbb{E}_{n^{(0:K)}}(f(x^{(k)})) - f(x^*)$$

$$f(x) - f(x^*) \underset{i.}{\leq} \nabla f(x)^T(x - x^*) \leq ||\nabla f(x)|| \, ||x - x^*|| \underset{iii.}{\leq} R||\nabla f(x)||$$

$$\mathbb{E}_{n^{(0:K)}}(||\nabla f(x^{(k)})||) \geq \frac{1}{R}\phi^{(k)} \quad (2)$$

Convergence / Proof (3/3)

$$\mathbb{E}_{n^{(k)}}(f(x^{(k+1)})) \underset{(1)}{\leq} f(x^{(k)}) - \frac{1}{2NL_{\max}}||\nabla f(x^{(k)})||^2 \quad |\mathbb{E}_{n^{(0:K)}}$$

$$\begin{aligned}
\phi^{(k+1)} &\leq \phi^{(k)} - \frac{1}{2NL_{\max}}\mathbb{E}_{n^{(0:K)}}(||\nabla f(x^{(k)})||^2) \\
&\underset{\text{Jensen's Ineq.}}{\leq} \phi^{(k)} - \frac{1}{2NL_{\max}}(\mathbb{E}_{n^{(0:K)}}(||\nabla f(x^{(k)})||))^2 \\
&\underset{(2)}{\leq} \phi^{(k)} - \frac{1}{2NL_{\max}R^2}(\phi^{(k)})^2 \qquad (3)
\end{aligned}$$

$$\frac{1}{\phi^{(k+1)}} - \frac{1}{\phi^{(k)}} = \frac{\phi^{(k)} - \phi^{(k+1)}}{\phi^{(k)}\phi^{(k+1)}} \geq \frac{\phi^{(k)} - \phi^{(k+1)}}{(\phi^{(k)})^2} \underset{(3)}{\geq} \frac{1}{2NL_{\max}R^2}$$

$$\frac{1}{\phi^{(k+1)}} \underset{\text{rec}}{\geq} \frac{1}{\phi^{(0)}} + \frac{k+1}{2NL_{\max}R^2} \geq \frac{k+1}{2NL_{\max}R^2}$$

Summary

- **Coordinate descent** minimizes one-dimensional subproblems for **a single variable x_n** at a time.
  - in cyclic or random order

- Coordinate descent can be fast if the **one-variable subproblems can be solved analytically**.

- If an x^* **minimizes all single-variable problems**, then it also is a minimizer for the all-variable problem
  - if f is convex and differentiable, or
  - if $f = g + h$ is the sum of
    - a convex and differentiable g and
    - a convex, possibly **non-differentiable**, but and **separable** $h(x_1, \ldots, x_N) = h_1(x_1) + h_2(x_2) + \cdots + h_N(x_N)$

- **Convergence** of coordinate descent can be proven for mild conditions
  - e.g., f differentiable, ∇f Lipschitz, sublevel set of $x^{(0)}$ bounded, and constant steplength $1/L_{\max}$.

Further Readings

- The coordinate descent method is not covered by Boyd and Vandenberghe [2004]
- Coordinate descent:
  - very briefly [Nocedal and Wright, 2006, ch. 9.3]
  - A brief, but dense survey: Wright [2015]
- Convergence proof: Wright [2015]

References

Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

Jorge Nocedal and Stephen J. Wright. *Numerical Optimization*. Springer Science+ Business Media, 2006.

Stephen J. Wright. Coordinate Descent Algorithms. *Math. Program.*, 151(1):3–34, June 2015. ISSN 0025-5610. doi: 10.1007/s10107-015-0892-3.

Outline

4. Examples

Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$

$$\min_{\mathbf{x}} \quad (a_1 x_1 - a_2 x_2 + a_3)^2$$

Algorithm:

- Initialize $\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}$
- Repeat until convergence:
  - $\mathbf{x}_1^{(k)} \leftarrow \arg\min_{x_1} (a_1 x_1 - a_2 \mathbf{x}_2^{(k-1)} + a_3)^2$
  - $\mathbf{x}_2^{(k)} \leftarrow \arg\min_{x_2} (a_1 \mathbf{x}_1^{(k)} - a_2 x_2 + a_3)^2$

Coordinate Descent - Example

$$\arg\min_{x_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Coordinate Descent - Example

$$\arg\min_{x_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_1}(a_1 x_1 - a_2 x_2 + a_3)^2 = 2(a_1 x_1 - a_2 x_2 + a_3)a_1$$

Coordinate Descent - Example

$$\arg\min_{x_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_1}(a_1 x_1 - a_2 x_2 + a_3)^2 = 2(a_1 x_1 - a_2 x_2 + a_3)a_1$$

$$x_1 = \frac{a_2 x_2 - a_3}{a_1}$$

Coordinate Descent - Example

$$\arg\min_{x_2}(a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_2}(a_1 x_1 - a_2 x_2 + a_3)^2$$

Coordinate Descent - Example

$$\arg\min_{x_2} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_2}(a_1 x_1 - a_2 x_2 + a_3)^2 = -2(a_1 x_1 - a_2 x_2 + a_3) a_2$$

Coordinate Descent - Example

$$\arg\min_{x_2} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_2}(a_1 x_1 - a_2 x_2 + a_3)^2 = -2(a_1 x_1 - a_2 x_2 + a_3)a_2$$

$$x_2 = \frac{a_1 x_1 + a_3}{a_2}$$

Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$

$$\min_{\mathbf{x}} \quad (a_1 x_1 - a_2 x_2 + a_3)^2$$

Algorithm:

- Initialize $\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}$
- Repeat until convergence:
  - $\mathbf{x}_1^{(k)} \leftarrow \frac{a_2 \mathbf{x}_2^{(k-1)} - a_3}{a_1}$
  - $\mathbf{x}_2^{(k)} \leftarrow \frac{a_1 \mathbf{x}_1^{(k)} + a_3}{a_2}$

Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$, $a_1 = 0.1$, $a_2 = 2$, $a_3 = 1$

$$\min_{\mathbf{x}} \quad (0.1x_1 - 2x_2 + 1)^2$$

$$\mathbf{x}_1^{(k)} \leftarrow \frac{2\mathbf{x}_2^{(k-1)} - 1}{0.1} \quad \mathbf{x}_2^{(k)} \leftarrow \frac{0.1\mathbf{x}_1^{(k)} + 1}{2}$$

Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$, $a_1 = 0.1$, $a_2 = 2$, $a_3 = 1$

$$\min_{\mathbf{x}} \quad (0.1x_1 - 2x_2 + 1)^2$$

$$\mathbf{x}_1^{(k)} \leftarrow \frac{2\mathbf{x}_2^{(k-1)} - 1}{0.1} \quad \mathbf{x}_2^{(k)} \leftarrow \frac{0.1\mathbf{x}_1^{(k)} + 1}{2}$$

Start with $\mathbf{x}_1^{(0)} = 1, \mathbf{x}_2^{(0)} = 2$

- $\mathbf{x}_1^{(1)} \leftarrow \frac{2 \cdot 2 - 1}{0.1} = 30$
- $\mathbf{x}_2^{(1)} \leftarrow \frac{0.1 \cdot 30 + 1}{2} = 2$

Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$, $a_1 = 0.1$, $a_2 = 2$, $a_3 = 1$

$$\min_{\mathbf{x}} \quad (0.1x_1 - 2x_2 + 1)^2$$

$$\mathbf{x}_1^{(k)} \leftarrow \frac{2\mathbf{x}_2^{(k-1)} - 1}{0.1} \quad \mathbf{x}_2^{(k)} \leftarrow \frac{0.1\mathbf{x}_1^{(k)} + 1}{2}$$

Start with $\mathbf{x}_1^{(0)} = 1, \mathbf{x}_2^{(0)} = 2$

- $\mathbf{x}_1^{(1)} \leftarrow \frac{2 \cdot 2 - 1}{0.1} = 30$
- $\mathbf{x}_2^{(1)} \leftarrow \frac{0.1 \cdot 30 + 1}{2} = 2$

$\mathbf{x}_1^{(1)} = 30, \mathbf{x}_2^{(1)} = 2$

- $\mathbf{x}_1^{(2)} \leftarrow \frac{2 \cdot 2 - 1}{0.1} = 30$
- $\mathbf{x}_2^{(2)} \leftarrow \frac{0.1 \cdot 30 + 1}{2} = 2$

Coordinate Descent for Linear Regression

For the problem

$$\text{minimize} \quad \sum_{i=1}^{m}(y_i - \mathbf{x}^T \mathbf{a_i})^2 = \sum_{i=1}^{m}\left(y_i - \sum_{j=1}^{n} x_j a_{ij}\right)^2$$

We can compute the update rule for a specific x_k:

$$\frac{\partial f(\mathbf{x})}{\partial x_k} \stackrel{!}{=} 0$$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 2 \cdot \sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij}\right)$$

Coordinate Descent for Linear Regression

$$2 \cdot \sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij} \right) = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1}^{n} x_j a_{ij} = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot \left(x_k a_{ik} + \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} \right) = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot x_k a_{ik} - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - x_k \sum_{i=1}^{m} a_{ik}^2 - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} = 0$$

Coordinate Descent for Linear Regression

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - x_k \sum_{i=1}^{m} a_{ik}^2 - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\, j\neq k}^{n} x_j a_{ij} = 0$$

$$x_k \cdot \sum_{i=1}^{m} a_{ik}^2 = \sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\, j\neq k}^{n} x_j a_{ij}$$

$$x_k = \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1,\, j\neq k}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2}$$

Linear Regression Coordinate Descent - Simple Algorithm

procedure LINEAR REGRESSION-CD
  input: f
  Get initial point $\mathbf{x}^{(0)}$
  repeat
    for $k \in 1, \ldots n$ **do**
      $$x_k \leftarrow \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1,\, j \neq k}^{n} x_j a_{ij}\right)}{\sum_{i=1}^{m} a_{ik}^2}$$
    end for
  until convergence
  return $\mathbf{x}$, $f(\mathbf{x})$
end procedure

Coordinate Descent for Linear Regression

For each parameter we have the following update rule:

$$x_k = \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2}$$

One Coordinate descent epoch has a cost of $O(m \cdot n^2)$!

Can we do it faster?

CD for Linear Regression - Smart Update

For each parameter we have the following update rule:

$$x_k = \frac{\sum_{i=1}^m a_{ik} \cdot \left(y_i - \sum_{j=1,\ j \neq k}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2}$$

We can rewrite:

$$\begin{aligned}
\sum_{i=1}^m \left(y_i - \sum_{j=1,\ j\neq k}^n x_j a_{ij} \right) &= \sum_{i=1}^m \left(y_i - \sum_{j=1}^n x_j a_{ij} + x_k a_{ik} \right) \\
&= \sum_{i=1}^m \left(y_i - \sum_{j=1}^n x_j a_{ij} \right) + \sum_{i=1}^m x_k a_{ik} \\
&= \sum_{i=1}^m \left(y_i - \sum_{j=1}^n x_j a_{ij} \right) + x_k \sum_{i=1}^m a_{ik}
\end{aligned}$$

CD for Linear Regression - Smart Update

From which we have:

$$x_k = \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1,\ j\neq k}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2}$$

$$= \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij} + \mathbf{x}_k^{old} a_{ik} \right)}{\sum_{i=1}^{m} a_{ik}^2}$$

$$= \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2} + \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(\mathbf{x}_k^{old} a_{ik} \right)}{\sum_{i=1}^{m} a_{ik}^2}$$

$$= \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2} + \frac{\mathbf{x}_k^{old} \cdot \sum_{i=1}^{m} a_{ik}^2}{\sum_{i=1}^{m} a_{ik}^2}$$

$$= \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2} + \mathbf{x}_k^{old}$$

CD for Linear Regression - Smart Update

Now we have:

$$x_k = \frac{\sum_{i=1}^m a_{ik} \cdot \left(y_i - \sum_{j=1}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2} + \mathbf{x}_k^{old}$$

So we can define our residual vector $\mathbf{r} \in \mathbb{R}^m$ such that

$$r_i = y_i - \sum_{j=1}^n x_j a_{ij}$$

CD for Linear Regression - Smart Update

After each update of x_k, we can maintain r_i instead of recomputing it given the old value $\mathbf{x}_k^{old}$:

$$\begin{aligned} r_i^{new} &= y_i - \left(\sum_{j=1}^{n} x_j a_{ij} - \mathbf{x}_k^{old} a_{ik} + x_k a_{ik} \right) \\ &= r_i^{old} + (\mathbf{x}_k^{old} - x_k) a_{ik} \end{aligned}$$

CD for Linear Regression - Smart Update

Now our algorithm looks like:

1. Initialize $\mathbf{x}$
2. Compute $r_i = y_i - \sum_{j=1}^{n} x_j a_{ij}$
3. While Not Converged
   3.1 For each $k = 1, \ldots, n$
      3.1.1 $\mathbf{x}_k^{old} \leftarrow x_k$
      3.1.2 $x_k \leftarrow \frac{\sum_{i=1}^{m} a_{ik} \cdot (r_i)}{\sum_{i=1}^{m} a_{ik}^2} + \mathbf{x}_k^{old}$
      3.1.3 For all i $r_i \leftarrow r_i + (\mathbf{x}_k^{old} - x_k) a_{ik}$

This algorithm is now $O(m \cdot n)$!

Linear Regression Coordinate Descent Algorithm

procedure LINEAR REGRESSION-CD
 input: f
 Get initial point $\mathbf{x}^{(0)}$
 $\mathbf{r} \leftarrow \mathbf{y} - A\mathbf{x}^{(0)}$
 repeat
  for $k \in 1, \ldots n$ **do**
   $\mathbf{x}_k^{old} \leftarrow x_k$
   $x_k \leftarrow \frac{\sum_{i=1}^m a_{ik} \cdot r_i}{\sum_{i=1}^m a_{ik}^2} + \mathbf{x}_k^{old}$
   for $i \in 1, \ldots m$ **do**
    $r_i \leftarrow r_i + (\mathbf{x}_k^{old} - x_k) a_{ik}$
   end for
  end for
 until convergence
 return $\mathbf{x}$, $f(\mathbf{x})$
end procedure

Real World Dataset: Body Fat prediction

We want to estimate the percentage of body fat based on various attributes:

- Age (years)
- Weight (lbs)
- Height (inches)
- Neck circumference (cm)
- Chest circumference (cm)
- Abdomen 2 circumference (cm)
- Hip circumference (cm)
- Thigh circumference (cm)

Real World Dataset: Body Fat prediction

The data is represented it as:

$$A_{m,n} = \begin{pmatrix} 1 & a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 1 & a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

with $m = 252$, $n = 14$

We can model the percentage of body fat y is a linear combination of the body measurements with parameters $\mathbf{x}$:

$$\hat{y}_i = \mathbf{x}^T \mathbf{a_i} = x_0 1 + x_1 a_{i,1} + x_2 a_{i,2} + \ldots + x_n a_{i,n}$$

Coordinate Descent - Body fat dataset

Year Prediction Data Set

- Least Squares Problem
- Prediction of the release year of a song from audio features
- 90 features
- Experiments done on a subset of 1000 instances of the data

Coordinate Descent - Year Prediction