Planning and Optimal Control

3. State Space Models

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Syllabus

Tue. 24.10.	(1)	1. Markov Models
Tue. 31.10.	—	— *Luther Day* —
Tue. 7.11	(2)	2. Hidden Markov Models
Tue. 14.11.	(3)	2b. (ctd.)
Tue. 21.11.	(4)	3. State Space Models
Tue. 28.11.	(5)	4. Markov Random Fields
Tue. 5.12.	(6)	5. Markov Decision Processes
Tue. 12.12.	(7)	6. Partially Observable Markov Decision Processes
Tue. 19.12.	(8)	
Tue. 26.12.	—	— *Christmas Break* —
Tue. 9.1.	(9)	7. Reinforcement Learning
Tue. 16.1.	(10)	
Tue. 23.1.	(11)	
Tue. 30.1.	(12)	
Tue. 6.2.	(13)	

Outline

1. Linear Gaussian Systems
2. State Space Models
3. Inference I: Kalman Filtering
4. Inference II: Kalman Smoothing
5. Learning via EM
6. Approximate Inference: Unscented Kalman Filter

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1. Linear Gaussian Systems

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6. Approximate Inference: Unscented Kalman Filter

Linear Transformation of a Gaussian

The linear transformation of a Gaussian is again a Gaussian:

$$
\begin{aligned}
p(x) &:= \mathcal{N}(x \mid \mu, \Sigma), && \mu \in \mathbb{R}^N, \Sigma \in \mathbb{R}^{N \times N} \\
y &:= Ax + a, && A \in \mathbb{R}^{M \times N}, a \in \mathbb{R}^M \\
\rightsquigarrow \quad p(y) &= p_y(Ax + a) = \mathcal{N}(y \mid A\mu + a, A\Sigma A^T)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\mathbb{E}(y) &= \mathbb{E}(Ax + a) = A\mathbb{E}(x) + a = A\mu + a \\
\mathbb{V}(y) &= \mathbb{E}((y - \mathbb{E}(y))(y - \mathbb{E}(y))^T) \\
&= \mathbb{E}(A(x - \mu)(A(x - \mu))^T) \\
&= A\mathbb{E}((x - \mu)(x - \mu)^T)A^T \\
&= A\Sigma A^T
\end{aligned}
$$

Product of two Gaussian PDFs

The product of two Gaussian PDFs is again Gaussian:

$$\mathcal{N}(x \mid \mu_1, \Sigma_1) \cdot \mathcal{N}(x \mid \mu_2, \Sigma_2) \propto \mathcal{N}(x \mid \mu, \Sigma)$$
$$\text{with} \quad \Sigma := (\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}$$
$$\mu := \Sigma(\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2)$$

Proof: elementary:

- $\log p$ is quadratic in x.
- complement squares.

Do not confuse this with

- $\mathcal{N}(x \mid \mu_1, \Sigma_1) \cdot \mathcal{N}(y \mid \mu_2, \Sigma_2) \propto \mathcal{N}(\begin{pmatrix} x \\ y \end{pmatrix} \mid \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix})$
- $p(x^2)$ for $x \sim \mathcal{N}(x \mid \mu, \Sigma)$.

Conditional Distributions of Multivariate Normals (Review)

Let y_A, y_B be jointly Gaussian

$$y := \begin{pmatrix} y_A \\ y_B \end{pmatrix} \sim \mathcal{N}(\begin{pmatrix} y_A \\ y_B \end{pmatrix} \mid \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix}, \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{pmatrix})$$

then the **conditional distribution** is

$$p(y_B \mid y_A) = \mathcal{N}(y_B \mid \mu_{B|A}, \Sigma_{B|A})$$

with

$$\mu_{B|A} := \mu_B + \Sigma_{BA}\Sigma_{AA}^{-1}(y_A - \mu_A)$$
$$\Sigma_{B|A} := \Sigma_{BB} - \Sigma_{BA}\Sigma_{AA}^{-1}\Sigma_{AB}$$

Conditional Distr. of Multiv. Normals / Information Form

Let y_A, y_B be jointly Gaussian

$$y := \begin{pmatrix} y_A \\ y_B \end{pmatrix} \sim \mathcal{N}(\begin{pmatrix} y_A \\ y_B \end{pmatrix} \mid \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix}, \Lambda = \begin{pmatrix} \Lambda_{AA} & \Lambda_{AB} \\ \Lambda_{BA} & \Lambda_{BB} \end{pmatrix})$$

then the **conditional distribution** is

$$p(y_B \mid y_A) = \mathcal{N}(y_B \mid \mu_{B|A}, \Lambda_{B|A})$$

with

$$\begin{aligned} \mu_{B|A} :=& \mu_B + \Lambda_{BB}^{-1}\Lambda_{BA}(y_A - \mu_A) \\ \Lambda_{B|A} :=& \Lambda_{BB}^{-1} \end{aligned}$$

Linear Gaussian System

$$p(x) := \mathcal{N}(x \mid \mu_x, \Sigma_x)$$
$$p(y \mid x) := \mathcal{N}(y \mid Ax + b, \Sigma_y)$$

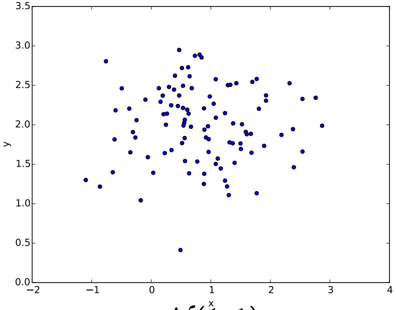
where

- x a multivariate Gaussian distributed random variable
  - $\mu_x \in \mathbb{R}^N, \Sigma_x \in \mathbb{R}^{N \times N}$
- y a multivariate Gaussian distributed random variable
  - $\mu_y := A\mu_x + b \in \mathbb{R}^M, \Sigma_y \in \mathbb{R}^{M \times M}$
  - $A \in \mathbb{R}^{M \times N}, b \in \mathbb{R}^M$
- y depends linearly on x

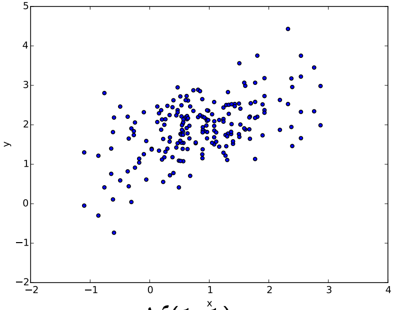
Linear Gaussian System

- LGS = multivariate multiple regression $(y|x)$ plus a Gaussian model for x.

- together, a generative Gaussian model.



$$x \sim \mathcal{N}(1, 1)$$
$$y \sim \mathcal{N}(2, 0.5)$$



$$x \sim \mathcal{N}(1, 1)$$
$$y \sim \mathcal{N}(x + 1, 0.5)$$

LGS as Joint Gaussian

An LGS

$$p(x) := \mathcal{N}(x \mid \mu_x, \Sigma_x)$$
$$p(y \mid x) := \mathcal{N}(y \mid Ax + b, \Sigma_y)$$

is equivalent to a jointly Gaussian distribution:

$$p(\begin{pmatrix} x \\ y \end{pmatrix}) = \mathcal{N}(\begin{pmatrix} \mu_x \\ A\mu_x + b \end{pmatrix}, \begin{pmatrix} \Sigma_x^{-1} + A^T\Sigma_y^{-1}A & -A^T\Sigma_y^{-1} \\ -\Sigma_y^{-1}A & \Sigma_y^{-1} \end{pmatrix}^{-1})$$

LGS as Joint Gaussian / Information Form

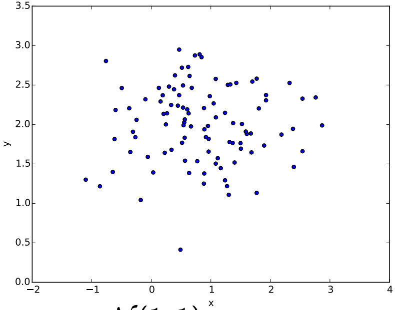
An LGS

$$p(x) := \mathcal{N}(x \mid \mu_x, \Lambda_x)$$
$$p(y \mid x) := \mathcal{N}(y \mid Ax + b, \Lambda_y)$$

is equivalent to a jointly Gaussian distribution:

$$p(\begin{pmatrix} x \\ y \end{pmatrix}) = \mathcal{N}(\begin{pmatrix} \mu_x \\ A\mu_x + b \end{pmatrix}, \begin{pmatrix} \Lambda_x + A^T \Lambda_y A & -A^T \Lambda_y \\ -\Lambda_y A & \Lambda_y \end{pmatrix})$$

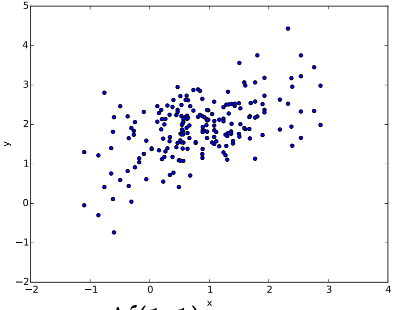
LGS as Joint Gaussian / Example



$$x \sim \mathcal{N}(1,1)$$
$$y \sim \mathcal{N}(2,0.5)$$

or equivalently

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\right)$$



$$x \sim \mathcal{N}(1,1)$$
$$y \sim \mathcal{N}(x+1,0.5)$$

or equivalently

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1.5 \end{pmatrix}\right)$$

Note: $\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 1.5 \end{pmatrix}$

LGS as Joint Gaussian / Proof

$$
\begin{aligned}
&\log p(x,y)\\
&= \log p(x) + \log p(y \mid x)\\
&\propto (x-\mu_x)^T \Lambda_x (x-\mu_x) + (y - Ax - b)^T \Lambda_y (y - Ax - b)\\
&= (x-\mu_x)^T \Lambda_x (x-\mu_x)\\
&\quad + (y - A\mu_x - b - A(x-\mu_x))^T \Lambda_y (y - A\mu_x - b - A(x-\mu_x))\\
&= (x-\mu_x)^T (\Lambda_x + A^T \Lambda_y A)(x-\mu_x)\\
&\quad + (y - A\mu_x - b)^T \Lambda_y (y - A\mu_x - b)\\
&\quad - 2(y - A\mu_x - b)^T \Lambda_y A (x-\mu_x)\\
&= \begin{pmatrix} x-\mu_x \\ y - A\mu_x - b \end{pmatrix}^T \begin{pmatrix} \Lambda_x + A^T\Lambda_y A & -A^T\Lambda_y \\ -\Lambda_y A & \Lambda_y \end{pmatrix} \begin{pmatrix} x-\mu_x \\ y - A\mu_x - b \end{pmatrix}
\end{aligned}
$$

Note: With $\Lambda_x := \Sigma_x^{-1}, \Lambda_y := \Sigma_y^{-1}$ precision matrices.

Bayes Rule for Linear Gaussian Systems

For an LGS

$$p(x) := \mathcal{N}(x \mid \mu_x, \Sigma_x)$$
$$p(y \mid x) := \mathcal{N}(y \mid Ax + b, \Sigma_y)$$

Bayes' Rule reads:

$$p(x \mid y) = \mathcal{N}(x \mid \mu_{x|y}, \Sigma_{x|y})$$
$$\text{with} \quad \Sigma_{x|y} := (\Sigma_x^{-1} + A^T \Sigma_y^{-1} A)^{-1}$$
$$\mu_{x|y} := \Sigma_{x|y} \left(A^T \Sigma_y^{-1} (y - b) + \Sigma_x^{-1} \mu_x \right)$$

Bayes Rule for Linear Gaussian Systems / Proof

- LGS is equivalent to joint Gaussian:

$$p(\begin{pmatrix} x \\ y \end{pmatrix}) = \mathcal{N}(\begin{pmatrix} \mu_x \\ A\mu_x + b \end{pmatrix}, \Lambda = \begin{pmatrix} \Lambda_x + A^T \Lambda_y A & A^T \Lambda_y \\ \Lambda_y A & \Lambda_y \end{pmatrix})$$

- conditional of a joint Gaussian:

$$p(x \mid y) = \mathcal{N}(x \mid \mu_{x|y}, \Lambda_{x|y})$$

with

$$\begin{aligned}
\Lambda_{x|y} :=& \Lambda_{x,x}^{-1} \\
\mu_{x|y} :=& \mu_x + \Lambda_{x,x}^{-1} \Lambda_{x,y}(y - \mu_y) \\
=& \Lambda_{x,x}^{-1}(\Lambda_{x,x}\mu_x + \Lambda_{x,y}(y - \mu_y)) \\
=& \Lambda_{x,x}^{-1}(\Lambda_x \mu_x + A^T \Lambda_y A \mu_x + A^T \Lambda_y (y - A\mu_x - b)) \\
=& \Lambda_{x,x}^{-1}(\Lambda_x \mu_x + A^T \Lambda_y (y - b))
\end{aligned}$$

Example: Inference from Noisy Measurements

- underlying quantity x
  - prior

$$p(x) := \mathcal{N}(x \mid \mu_x, \lambda_x^{-1})$$

- L noisy measurements $y_{1:L}$:

$$p(y_\ell \mid x) := \mathcal{N}(y_\ell \mid x, \lambda_y^{-1}), \quad \ell \in 1:L$$

  - scalar LGS: $N = M := 1$, $A := 1$ and $b := 0$: $\mu_y|x = Ax + b = x$
  - vector LGS: $N := 1, M := L$, $\mathbf{y} := y_{1:L}$, $\Lambda_y := \lambda_y \cdot I_{L\times L}$, $A := \mathbf{1}_L$, $\mathbf{b} := \mathbf{0}_L$,

$$\boldsymbol{\mu}_\mathbf{y}|\mathbf{x} = Ax + \mathbf{b} = x \cdot \mathbf{1}_L$$

Note: $I_{N\times N} := (\mathbb{I}(n = m))_{n,m\in 1:N}$ identity matrix.

Example: Inference from Noisy Measurements

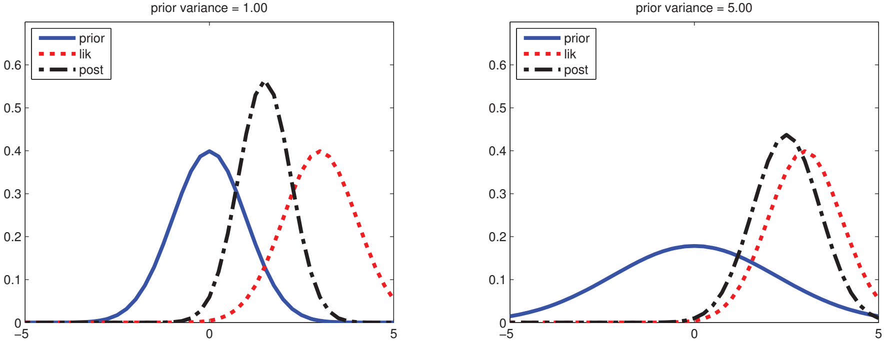
- vector LGS: $N = M := L$, $\mathbf{y} := y_{1:L}$, $\Lambda_y := \lambda_y \cdot I_{L \times L}$, $A := \mathbf{1}_L$, $\mathbf{b} := \mathbf{0}_L$,

$$\boldsymbol{\mu}_{\mathbf{y}}|\mathbf{x} = Ax + \mathbf{b} = x \cdot \mathbf{1}_L$$

- Bayes rule:

$$\begin{aligned}
p(x \mid y) &= \mathcal{N}(x \mid \mu_{x|y}, \Sigma_{x|y}) \\
\text{with} \quad \Sigma_{x|y}^{-1} &:= \Sigma_x^{-1} + A^T \Sigma_y^{-1} A \\
&= \lambda_x + L\lambda_y \\
\mu_{x|y} &:= \Sigma_{x|y} \left(A^T \Sigma_y^{-1} (y - b) + \Sigma_x^{-1} \mu_x \right) \\
&= (\lambda_x + L\lambda_y)^{-1} (\lambda_y \sum_{\ell=1}^{L} y_\ell + \lambda_x \mu_x) \\
&= \frac{\lambda_x}{\lambda_x + L\lambda_y} \mu_x + \frac{L\lambda_y}{\lambda_x + L\lambda_y} \frac{1}{L} \sum_{\ell=1}^{L} y_\ell
\end{aligned}$$

Example: Inference from Noisy Measurements



[source: Murphy 2012, p.121]

$$p(x) := \mathcal{N}(x \mid 0, \sigma^2 \in \{1,5\}), \quad p(y \mid x) := \mathcal{N}(y \mid x, 1), \qquad y = 3$$

prior: $p(x)$, MLE: $\mathcal{N}(x \mid y, 1)$, posterior: $p(x \mid y)$

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State Space Model

$z_t = g(z_{t-1})$	**transition model**
$x_t = h(z_t)$	**observation model**
$z_t \in \mathbb{R}^K$	**hidden state**
$x_t \in \mathbb{R}^M$	**observation**

- like HMM, but with continuous hidden state z_t
- g, h stochastic functions

Linear-Gaussian State Space Model

$p(z_t \mid z_{t-1}) := \mathcal{N}(z_t \mid A_t z_{t-1} + a_{t-1}, \Sigma_{z,t})$	**transition model**
$p(x_t \mid z_t) := \mathcal{N}(x_t \mid B_t z_t + b_t, \Sigma_{y,t})$	**observation model**
$z_t \in \mathbb{R}^K$	**hidden state**
$x_t \in \mathbb{R}^M$	**observation**
$A_t \in \mathbb{R}^{K \times K}$	transition matrix at time t
$B_t \in \mathbb{R}^{M \times K}$	observation matrix at time t
$\Sigma_{z,t} \in \mathbb{R}^{K \times K}$	state/system noise at time t
$\Sigma_{x,t} \in \mathbb{R}^{M \times M}$	observation noise at time t

- transition and observation function is linear
  - bias term often dropped: $a_{t-1} := 0$, $b_t := 0$.
- state and observation noise is Gaussian

Stationary Linear-Gaussian State Space Model

$p(z_t \mid z_{t-1}) := \mathcal{N}(z_t \mid Az_{t-1}, \Sigma_z)$	**transition model**
$p(x_t \mid z_t) := \mathcal{N}(x_t \mid Bz_t, \Sigma_y)$	**observation model**
$z_t \in \mathbb{R}^K$	**hidden state**
$x_t \in \mathbb{R}^M$	**observation**
$A \in \mathbb{R}^{K \times K}$	transition matrix
$B \in \mathbb{R}^{M \times K}$	observation matrix
$\Sigma_z \in \mathbb{R}^{K \times K}$	state/system noise
$\Sigma_x \in \mathbb{R}^{M \times M}$	observation noise

- **stationary**, **time-invariant**:
  - transition and observation matrices do not depend on time t

Initial State Distribution

All models need to be complemented by an **initial state distribution**:

$$p(z_1) := \mathcal{N}(z_1 \mid \mu_{z_1}, \Sigma_{z_1})$$

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Infering Posterior State Distributions $p(z_t \mid x_{1:t})$

Posterior hidden states can be computed sequentially:

$$p(z_t \mid x_{1:t}) = \mathcal{N}(z_t \mid \mu_t^\alpha, \Sigma_t^\alpha)$$

$$\text{with} \quad \Sigma_t^\alpha := ((A\Sigma_{t-1}^\alpha A^T)^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$

$$\mu_t^\alpha := \Sigma_t^\alpha ((A\Sigma_{t-1}^\alpha A^T)^{-1} A\mu_{t-1}^\alpha + B^T \Sigma_x^{-1} x_t)$$

$$\text{and} \quad \Sigma_1^\alpha := (\Sigma_{z_1}^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$

$$\mu_1^\alpha := \Sigma_1^\alpha (\Sigma_{z_1}^{-1} \mu_{z_1} + B^T \Sigma_x^{-1} x_1)$$

Infering $p(z_t \mid x_{1:t})$ / Proof

- for $t = 1$:

$$p(x_t \mid z_t) = \mathcal{N}(x_t \mid Bz_t, \Sigma_x)$$
$$p(z_1) = \mathcal{N}(z_t \mid \mu_{z_1}, \Sigma_{z_1})$$
$$\overset{\text{Bayes rule}}{\rightsquigarrow} \quad p(z_1 \mid x_1) = \mathcal{N}(z_t \mid \mu_1^\alpha, \Sigma_1^\alpha)$$
$$\text{with} \quad \Sigma_1^\alpha := \Sigma_{z_1|x_1} = (\Sigma_{z_1}^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$
$$\mu_1^\alpha := \mu_{z_1|x_1} = \Sigma_1^\alpha (\Sigma_{z_1}^{-1} \mu_{z_1} + B^T \Sigma_x^{-1} x_1)$$

- for $t > 1$:

$$p(x_t \mid z_t) = \mathcal{N}(x_t \mid Bz_t, \Sigma_x)$$
$$p(z_t \mid x_{1:t-1}) = \mathcal{N}(z_t \mid A\mu_{t-1}^\alpha, A\Sigma_{t-1}^\alpha A^T)$$
$$\overset{\text{Bayes rule}}{\rightsquigarrow} \quad p(z_1 \mid x_1) = \mathcal{N}(z_t \mid \mu_t^\alpha, \Sigma_t^\alpha)$$
$$\text{with} \quad \Sigma_t^\alpha := \Sigma_{z_t|x_{1:t}} = ((A\Sigma_{t-1}^\alpha A^T)^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$
$$\mu_t^\alpha := \mu_{z_t|x_{1:t}} = \Sigma_t^\alpha ((A\Sigma_{t-1}^\alpha A^T)^{-1} A\mu_{t-1}^\alpha + B^T \Sigma_x^{-1} x_t)$$

Precomputing Posterior Variances

- Σ_t^α does not depend on the observations $x_{1:t}$
  - thus can be precomputed
- Σ_t^α depends on t only through the time since the initial state
  - if we assume states long after the initial state, use

$$\Sigma^\alpha := \lim_{t\to\infty} \Sigma_t^\alpha$$

    for all t.

  - Σ^α can be computed via fixpoint iterations

$$(\Sigma^\alpha)^{(0)} := (\Sigma_{z_1}^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$
$$(\Sigma^\alpha)^{(t)} := ((A(\Sigma^\alpha)^{(t-1)} A^T)^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$

Computing Variances with a Single Matrix Inversion

- in its previous form, computing variances Σ_t^α requires two matrix inversions:

$$\Sigma_t^\alpha := ((A\Sigma_{t-1}^\alpha A^T)^{-1} + B^T \Sigma_x^{-1} B)^{-1}$$

- more efficient computation with a single matrix inversion:

$$\begin{aligned}\Sigma_{t|t-1} &:= A\Sigma_{t-1}^\alpha A^T \\ \Sigma_t^\alpha &:= (I - \underbrace{\Sigma_{t|t-1} B^T (\Sigma_x + B\Sigma_{t|t-1} B^T)^{-1}}_{=:K_t} B)\Sigma_{t|t-1} \\ &= (I - K_t B)\Sigma_{t|t-1}\end{aligned}$$

Proof: apply the matrix inversion lemma

$$(A - BD^{-1}C)^{-1} = (I + A^{-1}B(D - CA^{-1}B)^{-1}C)A^{-1}$$

to $(\Sigma_{t|t-1}^{-1} + B^T \Sigma_x^{-1} B)^{-1}$

Computing Means without Additional Matrix Inversion

- also the original mean formula contains a matrix inversion:

$$\mu_t^\alpha := \Sigma_t^\alpha (B^T \Sigma_x^{-1} x_t + \Sigma_{t|t-1}^{-1} A \mu_{t-1}^\alpha)$$

- can be simplified, reusing the matrix inversion from the variance:

$$\mu_{t|t-1} := A\mu_{t-1}^\alpha$$
$$\mu_t^\alpha = \mu_{t|t-1} + K_t(x_t - B\mu_{t|t-1})$$

proof:

left term: using 2nd matrix inversion fomula

$$\Sigma_t^\alpha B^T \Sigma_x^{-1} = \Sigma_{t|t-1} B^T (\Sigma_x + B\Sigma_{t|t-1} B^T)^{-1} = K_t$$
$$(A - BD^{-1}C)^{-1} BD^{-1} = A^{-1} B (D - CA^{-1}B)^{-1}$$

right term:

$$\Sigma_t^\alpha \Sigma_{t|t-1}^{-1} = (I - K_t B)\Sigma_{t|t-1} \Sigma_{t|t-1}^{-1} = (I - K_t B)$$

Kalman Filtering (Single Inversion)

- prediction step:

$$\Sigma_{t|t-1} := A\Sigma^{\alpha}_{t-1}A^T$$
$$\mu_{t|t-1} := A\mu^{\alpha}_{t-1}$$

- measurement step:

$$K_t := \Sigma_{t|t-1}B^T(\Sigma_x + B\Sigma_{t|t-1}B^T)^{-1}$$
$$\mu^{\alpha}_t = \mu_{t|t-1} + K_t(x_t - B\mu_{t|t-1})$$
$$\Sigma^{\alpha}_t := (I - K_tB)\Sigma_{t|t-1}$$

Kalman Filtering / Algorithm

infer-filtering-kalman$(x, A, \Sigma_z, B, \Sigma_x, \mu_{z_1}, \Sigma_{z_1})$:

  $T := |x|$

  $\Sigma_1^\alpha := (\Sigma_{z_1}^{-1} + B^T \Sigma_x^{-1} B)^{-1}$

  $\mu_1^\alpha := \Sigma_1^\alpha (B^T \Sigma_x^{-1} x_1 + \Sigma_{z_1}^{-1} \mu_{z_1})$

  for $t = 2, \ldots, T$:

    $\Sigma_{t|t-1} := A \Sigma_{t-1}^\alpha A^T$

    $\mu_{t|t-1} := A \mu_{t-1}^\alpha$

    $K_t := \Sigma_{t|t-1} B^T (\Sigma_x + B \Sigma_{t|t-1} B^T)^{-1}$

    $\mu_t^\alpha = \mu_{t|t-1} + K_t (x_t - B \mu_{t|t-1})$

    $\Sigma_t^\alpha := (I - K_t B) \Sigma_{t|t-1}$

  return $\mu_{1:T}^\alpha, \Sigma_{1:T}^\alpha$

where

- $x \in (\mathbb{R}^M)^*$ observed sequence
- $A, \Sigma_z, B, \Sigma_x, \mu_{z_1}, \Sigma_{z_1}$ linear-Gaussian state space model

yields $p(z_t \mid x_{1:t}) = \mathcal{N}(z_t \mid \mu_t^\alpha, \Sigma_t^\alpha), t = 1 : T$ PDFs of filtered latent states

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Infering Posterior State Distributions $p(z_t \mid x_{1:T})$

$$p(z_t \mid x_{1:T}) = \mathcal{N}(z_t \mid \mu_t^\gamma, \Sigma_t^\gamma)$$
$$\mu_t^\gamma := \mu_t^\alpha + J_t(\mu_{t+1}^\gamma - \mu_{t+1|t})$$
$$\Sigma_t^\gamma := \Sigma_t^\alpha + J_t(\Sigma_{t+1}^\gamma - \Sigma_{t+1|t})J_t^T$$
$$J_t := \Sigma_t^\alpha A^T \Sigma_{t+1|t}$$

backwards Kalman gain matrix

with

$$p(z_{t+1} \mid x_{1:t}) = \mathcal{N}(z_t \mid \mu_{t+1|t}, \Sigma_{t+1|t})$$

prediction

$$\mu_{t+1|t} = A\mu_t^\alpha$$
$$\Sigma_{t+1|t} = A\Sigma_t^\alpha A^T + \Sigma_x$$

initialized by $p(z_T \mid x_{1:T})$, i.e.,

$$\mu_T^\gamma := \mu_T^\alpha, \quad \Sigma_T^\gamma := \Sigma_T^\alpha$$

Infering Posterior State Distr. $p(z_t \mid x_{1:T})$ / Proof

$$p(z_t \mid x_{1:T}) = \int_{z_{t+1}} p(z_{t+1} \mid x_{1:T})\, p(z_t \mid x_{1:t}, \cancel{x_{t+1:T}}, z_{t+1}) dz_{t+1}$$

$$p(z_t, z_{t+1} \mid x_{1:t}) = \mathcal{N}(\begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix} \mid \begin{pmatrix} \mu_t^\alpha \\ \mu_{t+1|t} \end{pmatrix}, \begin{pmatrix} \Sigma_t^\alpha & \Sigma_t^\alpha A^T \\ A\Sigma_t^\alpha & \Sigma_{t+1|t} \end{pmatrix})$$

filtered two-slice posteriors

Gaussian conditioning yields

$$p(z_t \mid x_{1:t}, z_{t+1}) = \mathcal{N}(z_t \mid \mu_t^\alpha + J_t(z_{t+1} - \mu_{t+1|t}), \Sigma_t^\alpha - J_t \Sigma_{t+1|t} J_t^T)$$

and finally

$$\begin{aligned} \mu_t^\gamma &= \mathbb{E}(\mathbb{E}(z_t \mid z_{t+1}, x_{1:T}) \mid x_{1:T}) \\ &= \mathbb{E}(\mathbb{E}(z_t \mid z_{t+1}, x_{1:t}) \mid x_{1:T}) \\ &= \mathbb{E}(\mu_t^\alpha + J_t(z_{t+1} - \mu_{t+1|t}) \mid x_{1:T}) \\ &= \mu_t^\alpha + J_t(\mu_{t+1}^\gamma - \mu_{t+1|t}) \end{aligned}$$

Infering Posterior State Distr. $p(z_t \mid x_{1:T})$ / Proof

$$\begin{aligned}
\Sigma_t^\gamma &= \mathbb{V}(\mathbb{E}(z_t \mid z_{t+1}, x_{1:T}) \mid x_{1:T}) + \mathbb{E}(\mathbb{V}(z_t \mid z_{t+1}, x_{1:T}) \mid x_{1:T}) \\
&= \ldots \\
&= \Sigma_t^\alpha + J_t(\Sigma_{t+1}^\gamma - \Sigma_{t+1|t})J_t^T
\end{aligned}$$

Outline

1. Linear Gaussian Systems

2. State Space Models

3. Inference I: Kalman Filtering

4. Inference II: Kalman Smoothing

5. Learning via EM

6. Approximate Inference: Unscented Kalman Filter

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1. Linear Gaussian Systems
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Summary

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Further Readings

- Inference in jointly Gaussian distributions:
  - lecture Machine Learning 2, ch. A.2 Gaussian Processes
  - Murphy 2012, chapter 4.3.
- Linear Gaussian Systems:
  Murphy 2012, chapter 4.4.
- State Space Models:
  Murphy 2012, chapter 18.

References

Kevin P. Murphy. *Machine Learning: A Probabilistic Perspective*. The MIT Press, 2012.