Topological Relations from Metric Refinements

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ABSTRACT
Naive Geography's premise "Topology matters, metric refines" calls for metric properties that provide opportunities for finer-grained distinctions than the purely qualitative topological relations. This paper defines a comprehensive set of eleven metric refinements that apply to the eight coarse topological relations between two regions that the 9-intersection and the Region-Connection Calculus identify and develops the applicable value ranges for each metric refinement. It is shown that any topological relation between two regions can be derived uniquely from the conjunction of at most three such refinement specifications (i.e., pairs of metric refinements and applicable value ranges). The smallest set of refinement specifications that determine uniquely all eight relations resorts to six of the eleven metric refinements.

Categories and Subject Descriptors
H.2.8 [**Database Applications**]: Spatial databases and GIS

General Terms
Management, Design.

Keywords
Spatial reasoning, metric, spatial relations, topology.

1. INTRODUCTION
The mathematical concepts of *topology* and *metric* have had a profound impact on the modeling of discrete spatial phenomena. While topology addresses invariants under homeomorphisms (i.e., properties that do not rely on metric measures), a metric space is based on the notion of a distance. Topology and metric are not totally separate concepts, however. Topology and metric both establish a closeness, and a metric space induces a topology. A metric space, such as $\mathbb{R}^2$ or $\mathbb{Z}^2$, therefore, exhibits both metric and topological properties that are interlinked. For qualitative spatial reasoning, however, this linkage between topology and metric has not yet been established sufficiently.

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In geographic information systems the rise of topological data models and data structures was to a large degree driven by the unreliable results of geometric calculations with finite number systems [13]. The same computational deficiencies of numerical calculations stimulated Naive Physics [16], calling for alternatives to traditional Euclidean geometry. These concerns about numerical inconsistencies have taken the backseat over time, however. More recently the prime motivation for topological models has been the quest for *cognitively plausible* qualitative spatial models. Since topological spaces are more abstract than metric spaces, topological properties have assumed a leading role in qualitative spatial reasoning. In this setting, the 9-intersection [9] and the Region-Connection-Calculus RCC [21] for binary topological relations have become popular models for qualitative spatial relations. Most models for qualitative distances [15, 17, 22, 25] have been developed independently of topological models, although some approaches have considered metric refinements of topological relations [5, 11, 20, 23] including specific shapes, such as convex hulls [2]. Here we are interested primarily in the reverse dependencies—that is, from metric to topology—and their implications. Such cross-relation dependencies have been found useful for the combination of topological and direction relations [24] yielding at times unique inferences where topology or direction alone would only provide ambiguous results. On the other hand reasoning about distances alone, without considerations of directions, is often inconclusive [12]. Likewise combined reasoning about topological relations and size of regions has been explored [16, 18].

A key question is whether the entire set of eight topological relations between two spatial regions can be derived uniquely from a combination of metric properties. Such inferences would contribute to generating approximate graphical depictions from verbal descriptions. They would also lay the foundation for combined metric and topological reasoning, particularly if topological information is enhanced by qualitative metric information.

The remainder of the paper is structured as follows: Section 2 briefly reviews the underlying model for topological relations and introduces terminology referred to later. Section 3 introduces the systematic model for metric refinements. The analysis of the value ranges of the eleven metric refinements leads to refinement specifications (Section 4), from which topological relations are inferred (Section 5). The paper closes with conclusions and a discussion of future work (Section 6).

2. TOPOLOGICAL RELATIONS
The spatial objects considered in this paper, referred to as *spatial regions*, are embedded in $\mathbb{R}^2$. They have a continuous boundary, no holes, no spikes, and no cuts. Whenever two

such spatial regions are considered, one is labeled A and the other B. Every spatial region has three distinct features: (1) an *interior* (denoted as A°), (2) a *boundary* (denoted as ∂A), and (3) an *exterior* (denoted as A^-). A° is the union of all open sets contained in A, ∂A is the difference of the closure—the intersection of all closed sets containing A—minus A°, and A^- is the difference of A's embedding space and A's closure.

The topological relations between two such spatial regions have been derived based on the 9-intersection matrices of the two regions' interiors, boundaries, and exteriors [9]. For an embedding in $\mathbb{R}^2$ eight 3x3 matrices with empty and non-empty values, called the *content invariant*, apply (Figure 1), each yielding a different topological relation. This set of relations is jointly exhaustive and pairwise disjoint, so that all combinations of two objects exhibit one and only one of the spatial relations.

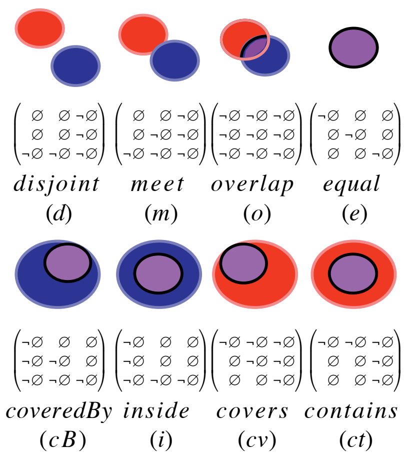


Figure 1. The eight topological relations between two regions in $\mathbb{R}^2$ with their 9-intersection matrices, the relations' labels, and their shortcuts.

The content invariant provides only a coarse classification of topological relations. While empty intersections offer no further opportunities to distinguish more detail, any non-empty intersection has the potential to capture a plethora of more detailed properties. Within this framework, a series of topological refinements of non-empty intersection have been identified, such as the dimension of an intersection and the number of separations [4], or for non-empty intersections involving lines the types of the intersection [1, 8].

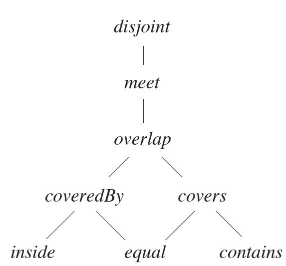


Figure. 2. The conceptual A-neighborhood of the eight topological relations between two regions in $\mathbb{R}^2$.

Similarity among these qualitative relations is captured through their conceptual neighborhoods [6, 14], which link relations that can be obtained through a topological transformation without a need to go through another relation. The A-neighborhood of the eight region-region relations (Figure 2) is most fundamental as it is established through purely topological means.

3. METRIC REFINEMENTS FOR REGION-REGION RELATIONS

The premise of metric refinements is to enhance topological relations with non-topological discernability. The metric refinements of the topological relations distinguish two types of measures, those that record metric properties of common parts between two regions and those that capture how far non-intersecting parts are from either other. In terms of the 9-intersection matrices this distinction translates to the following metric analysis of non-empty intersections for region-region relations through *splitting measures*:

- the intersection's area for a 2-dimensional intersection
- the intersection's length for a 1-dimensional intersection
- 0 for an intersection that consists only of 0-dimensional components

Similarly, *closeness measures* apply to empty intersections of two linear objects, such as two regions' boundaries. These metric refinements materialize in terms of the area of a buffer zone, a fundamental GIS operation.

In order to yield scale-independent values these areas and distances are normalized by the reference region's interior area and its perimeter (i.e., the boundary's length), yielding a dimension-neutral metric for each measure.

3.1 SPLITTING MEASURES

Each splitting measure is defined as a ratio, either of the area of an intersection with respect to the area of one of the two regions, or of the length of an intersection with respect to the length of the boundary of one of the two regions. Among the nine intersections of interiors, boundaries, and exteriors, four are two-dimensional ($A^\circ \cap B^\circ, A^\circ \cap B^-, A^- \cap B^\circ$, and $A^- \cap B^-$) and, therefore, of type area, while another four are linear ($A^\circ \cap \partial B, \partial A \cap B^\circ, \partial A \cap B^-$, and $A^- \cap \partial B$) and, therefore, of type length. The ninth intersection ($\partial A \cap \partial B$) may be linear or 0-dimensional, depending on the particular geometry of the boundaries' intersection. Their notions and informal definitions are complemented by the refinements formal definitions and graphical examples highlighting their components (Figure 3).

- Inner Area Splitting (IAS): the portion of A's interior inside of B (Figure 3a).
- Outer Area Splitting (OAS): the portion of A's interior outside of B (Figure 3c).
- Inverse Outer Area Splitting (OAS^{-1}): the portion of A's exterior inside of B (Figure 3g).

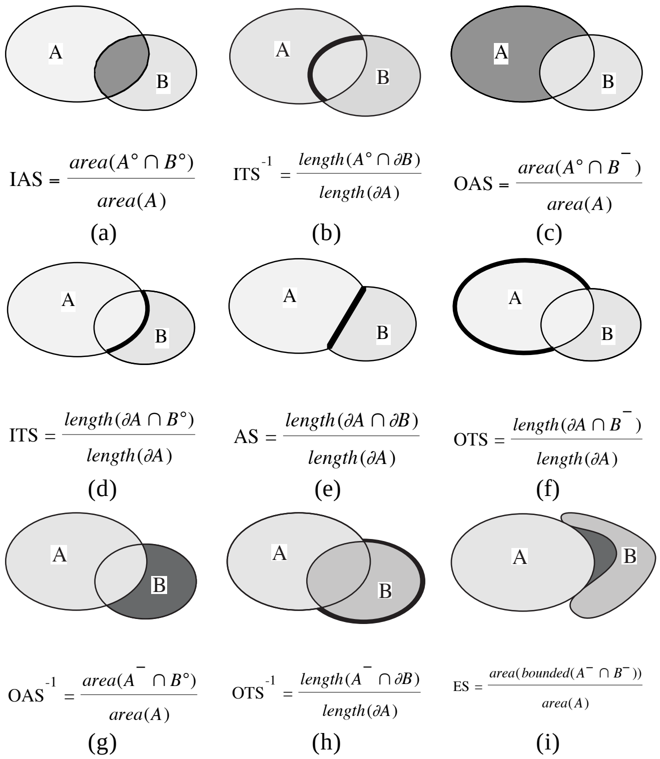


Figure 3. The nine splitting measures.

- Exterior Splitting (ES): the area of A's exterior shut off by the union of A and B (Figure 3i).
- Inner Traversal Splitting (ITS): the portion of A's boundary inside of B (Figure 3d).
- Inverse Inner Traversal Splitting (ITS^{-1}): the portion of A's interior shared with B's boundary (Figure 3b).
- Outer Traversal Splitting (OTS): the portion of A's boundary outside of B (Figure 3f).
- Inverse Outer Traversal Splitting (OTS^{-1}): the portion of A's exterior shared with B's boundary (Figure 3h).
- Alongness Splitting (AS): the portion of A's boundary shared with B (Figure 3e).

Together these nine splitting measures offer a refinement opportunity for the entire 9-intersection (Figure 4).

	B°	∂B	B^-
A°	IAS	ITS^{-1}	OAS
∂A	ITS	AS	OTS
A^-	OAS^{-1}	OTS^{-1}	ES

Figure 4. Distribution of splitting measures over 9-intersection

Since IAS and OAS are based on $A^\circ \cap B^\circ$ and $A^\circ \cap B^-$, respectively, and both measures are normalized by the same value, the two measures exhaust the entire area of region A. This implies that IAS and OAS are complementary so that they must sum up to 1 (Equation 1).

$$\text{IAS} + \text{OAS} = 1 \tag{1}$$

The splitting measures IAS and OAS^{-1}, with $A^\circ \cap B^\circ$ and $A^- \cap B^\circ$, extend similarly exhaustively over region B. The normalization by A's area, however, yields only the conclusion that IAS and OAS^{-1} are complementary, but provides no insight about their sum. No immediate dependencies can be found for the fourth areal measure ES.

The traversal splittings reveal another dependency. The three measures ITS, AS, and OTS, with $\partial A \cap B^\circ$, $\partial A \cap \partial B$, and $\partial A \cap B^-$, extend exhaustively over A's boundary. Since each is normalized by A's perimeter, their sum must yield 1 (Equation 2).

$$\text{ITS} + \text{OTS} + \text{AS} = 1 \tag{2}$$

The traversal measures ITS^{-1}, AS, and OTS^{-1} then expose the analog of IAS and OAS^{-1} since their sum covers completely B's boundary, but the normalization by A's perimeter only implies a dependency among the three values with respect to B's perimeter, but they do not yield a specific sum.

3.2 CLOSENESS MEASURES

Closeness measures capture the effort to convert an empty boundary-boundary intersection into a non-empty intersection. They are defined as the area by which a region needs to grow or shrink in order for its boundary to make contact with the other region.

- Expansion Closeness (EC): the swelling required for A and B so that their boundaries intersect (Figure 5a).
- Contraction Closeness (CC): the contraction required for A and B so that their boundaries intersect (Figure 5b).

For Contraction Closeness, the area of the reference region normalizes this buffer zone to a value between 0 and 1, while for the Expansion Closeness the buffer is normalized by the area after swelling (i.e., the area of the union of the reference region *and* the buffer). The specifications of buffer zones for metric details of line-line relations [20] apply.

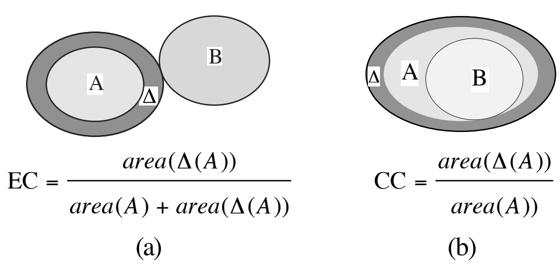


Figure. 5. The two closeness measures (a) Expansion Closeness (EC) and (b) Contraction Closeness (CC).

4. REFINEMENT SPECIFICATIONS

For different topological relations these metric refinements carry different values, which may become characteristics to identify topological relations from metric properties. For example, the IAS value for *disjoint* is 0, because $A^\circ \cap B^\circ = \emptyset$. The combination of a metric refinement and such a value is called a *refinement specification*. In this section the refinement specifications for all eight topological relations are derived from the refinements' equations (Figures 3 and 5). Since for all metric refinements the denominator is greater than zero, no division by zero may occur. All refinement specifications are summarized graphically (Section 4.9) to enable a visual comparison.

4.1 disjoint

Two *disjoint* regions have no common interiors or boundaries, therefore, the numerators of IAS, ITS^{-1}, ITS, and AS are zero. The corresponding denominators are non-zero, implying also zero values for these metric refinements.

Since two *disjoint* regions are separated they cannot form a bounded exterior so that ES must be zero as well. The regions' separation also implies that the reference region's interior and boundary are completely located in the target region's exterior, so that OAS and OTS become divisions of identical areas and identical perimeters, yielding values of 1 for both.

The precise values of the two inverse outer measures OAS^{-1} and OTS^{-1} depend on the metric properties relative to the target region's interior and boundary, so that their values may extend from >0 to +∞.

The Expansion Closeness EC for disjoint is based on a greater-than-zero buffer zone, which implies a value range between greater than zero and less than 1. Due to the *disjoint* regions being located in their opposite regions' exteriors the buffer zone for contraction closeness is zero, implying CC=0.

4.2 meet

The relation *meet* exposes most of the same properties as *disjoint*, except for those that are related to *meet*'s common boundary parts. So *meet* shares with *disjoint* IAS=0, ITS^{-1}, OAS=1, ITS=0, OAS^{-1}>0, OTS^{-1}>0, and CC=0.

The common boundary may be a single point or a sequence of separated point segments [8], which yield an Alongness Splitting value of 0. In these cases the length of the boundary located in that region's exterior is equal to the entire perimeter, implying that OTS is 1.

For scenarios in which the common boundary consists of at least one non-point segment the Alongness Splitting is greater than 0, but it can never reach one, which would require full coincidence between the two boundaries. So *meet*'s value range for AS is the open interval (0 1). Its immediate implication for OTS is that OTS's value range must then be (0 1) as well. Therefore *meet* yields two value ranges each for AS and OTS.

Whenever the boundary-boundary intersection has a separation, the two regions that *meet* form one or more bounded areas of the exterior. With a non-zero total of these bounded areas the Exterior Splitting assumes as value of greater than zero. Whenever meet has a single boundary component, however, there is no separation of the exterior so that ES becomes 0.

The last value range that differs between *disjoint* and *meet* is for Expansion Closeness, since *meet*'s boundary-boundary intersection is non-empty, which implies a zero-buffer zone for EC, which also yields an EC value of zero.

4.3 overlap

For overlap all nine intersection are non-empty so that all splitting measures apply, while both closeness measures EC and CC are 0.

For IAS, OAS, ITS, and OTS the common part may be any non-zero subset of the reference region's interior or boundary, but not the entire interior or boundary, because otherwise a different topological relation would hold. Therefore, each of IAS, OAS, ITS, and OTS must be greater than 0 but less than 1.

For the inverse splittings the common perimeter or area may be less than or greater than the area or perimeter of the target region, but never 0. This implies that for *overlap* ITS^{-1} and OTS^{-1} must be greater than 0.

With respect to Exterior Splitting *overlap* has the same setting as *meet*, because the two regions' boundaries may create one or more bounded portions of the exterior [8], so that ES may be greater than 0. It is, however, also feasible that overlap forces no separation of the exterior so that ES may remain 0.

The outer inverse splittings OAS-1 and OTS-1 have in the numerator a non-empty part of the target region's interior or boundary, respectively. Their values are, therefore, the positive real numbers.

4.4 equal

The coincidence of interiors, boundaries, and exteriors for *equal* constrains its metric refinements so that no value ranges are possible. Its six empty intersections imply zero values for IAS, ITS^{-1}, OAS, ITS, OTS, OAS^{-1}, and OTS^{-1}. IAS and AS respectively derive from the intersections of the entire interior and entire boundary, yielding the values 1. The third non-empty interior—exterior-exterior—cannot form a bounded separation so that ES is always 0. Since the boundaries coincide there are no opportunities for forming non-zero buffer zones, which implies that EC and CC are zero.

4.5 coveredBy

The three empty intersections of *coveredBy* imply zero values for their corresponding metric refinements ITS^{-1}, OAS, and OTS. Since the non-empty boundary-boundary intersection yields no opportunities for forming non-zero buffer zones, which implies that EC and CC are zero as well. While two foundations for Exterior Splitting are fulfilled—boundary-boundary and interior-interior intersections are non-empty—a third constraint—that both boundaries must intersect their opposite exteriors—is not fulfilled [3], therefore, no bounded part of the exterior-exterior intersection can be formed for *coveredBy*, which implies ES=0. *A coveredBy B* implies $A^\circ \subset B^\circ$ so that IAS is the division of the areas of A° by *A*, yielding 1. The true subset constraint $A \subset \overline{B}$ of *coveredBy* implies that non-empty portions of *B*'s interior and *B*'s boundary are located in *A*'s exterior. Therefore, the two inverse outer splittings must be greater than 0.

The assessment of the traversal splittings ITS and AS follows the analogous discussion of the *meet*'s values for OTS and AS. If the common boundary of *coveredBy* consists only of one or more separated point-like segments, then the length of the boundary-boundary intersection is 0, which implies AS=0 and since OTS=0, it follows (Equation 2) that ITS=1. Conversely, for at least one non-point segment the Alongness Splitting is greater than 0, but it can never reach one (which would require full coincidence between the two boundaries). 0<AS<1 must then be paired with 0<ITS<1.

Since two *disjoint* regions are separated they cannot form a bounded exterior so that ES must be zero as well. The regions' separation also implies that the reference region's interior and boundary are completely located in the target region's exterior, so that OAS and OTS become divisions of identical areas and identical perimeters, yielding values of 1 for both.

The precise values of the two inverse outer measures OAS^{-1} and OTS^{-1} depend on the metric properties relative to the target region's interior and boundary, so that their values may extend from >0 to +∞.

The Expansion Closeness EC for disjoint is based on a greater-than-zero buffer zone, which implies a value range between greater than zero and less than 1. Due to the *disjoint* regions being located in their opposite regions' exteriors the buffer zone for contraction closeness is zero, implying CC=0.

4.6 *covers*

Since *coveredBy* and *covers* are pairs of converse relations, the refinement specifications of *covers* can be derived directly from those of *coveredBy*. ITS, OAS^{-1}, and OTS^{-1} must be 0 due to the emptiness of their corresponding intersections. The empty values of ES, EC, and CC for *coveredBy* apply unchanged to *covers*, while complementary properties lead to 0<IAS<1, ITS-1>0, and 0<OAS<1 for *coveredBy*. For the three traversal splittings ITS and OTS switch specifications, so that for *coveredBy* OTS=1 with AS=0, while 0<OTS<1 pairs with 0<AS<1.

4.7 *inside*

Since *inside* differs from *coveredBy* only the value of the boundary-boundary intersection, they share all splitting values except for Along Splitting, which is empty for *inside* and, therefore, must be 0, and ITS, which is fixed at 1 for *inside*. While the Contraction Closeness does not apply (therefore, CC=0), a non-empty buffer can be formed around the contained region *A* so that neighboring relations *coveredBy* or *equal* are obtained. Such a non-empty buffer leads to a non-zero numerator in EC. The buffer's area is, however, always less than EC denominator—the area of the buffer plus the area of the reference region—so that 0<EC<1.

4.8 *contains*

The refinement specifications for *contains* can be extrapolated in two ways: (1) making in analogy to the transition from *coveredBy* to *inside* the transition form *covers* to *contains*; or (2) following the converseness reasoning from *coveredBy* to *covers* and applying it to the transition from *inside* to *contains*. In either case, ITS, AS, OAS^{-1}, and OTS^{-1} are 0 since their corresponding intersections are empty. Since ITS=0 and AS=0, OTS must be 1 (Equation 2). No Exterior Splitting applies (i.e., ES=0). IAS and OAS are both in the range between 0 and 1, while $ITS^{-1}>0$. Since $A \supset \overline{B}$, no expansion buffer applies to *A*, so that EC=0. A contraction buffer, however, can be formed to its neighboring relations *covers* and *equal*. That buffer's area is greater than zero, but cannot reach the size of *A*. Therefore, 0<CC<1.

4.9 Summary

The mappings from the topological relations onto the refinement specifications (Figure 6) provide for each refinement measure a graphical account of what values or value ranges apply to what topological relation. Seven value cases occur: The two values 0 and 1, three interval ranges between 0 and 1—the open interval (0 1) and the two semi-open intervals [0 1) and (0 1]—and two ranges whose upper bounds exceed 1—the positive real numbers (0 ∞) and the non-negative real numbers [0 ∞). The topological relations' conceptual neighborhood graph [6] frames this portrayal. Each metric refinement has between 2 (Figures 6b and 6g-k) and 4 (Figure 6f) value ranges. Common value ranges form clusters that distribute over the neighborhood graph. For some refinements each range of common values is connected (Figures 6a, c, g-h). Since some pairs of the seven value ranges are not mutually exclusive, only clusters involving mutually exclusive value ranges contribute to the formation of simply connected clusters. The study of the distribution of value ranges over topological neighborhoods, the combination of the value ranges' neighborhoods with the topological relations' neighborhoods, and any differences from type A, B, or C neighborhoods [14] are outside this paper's scope.

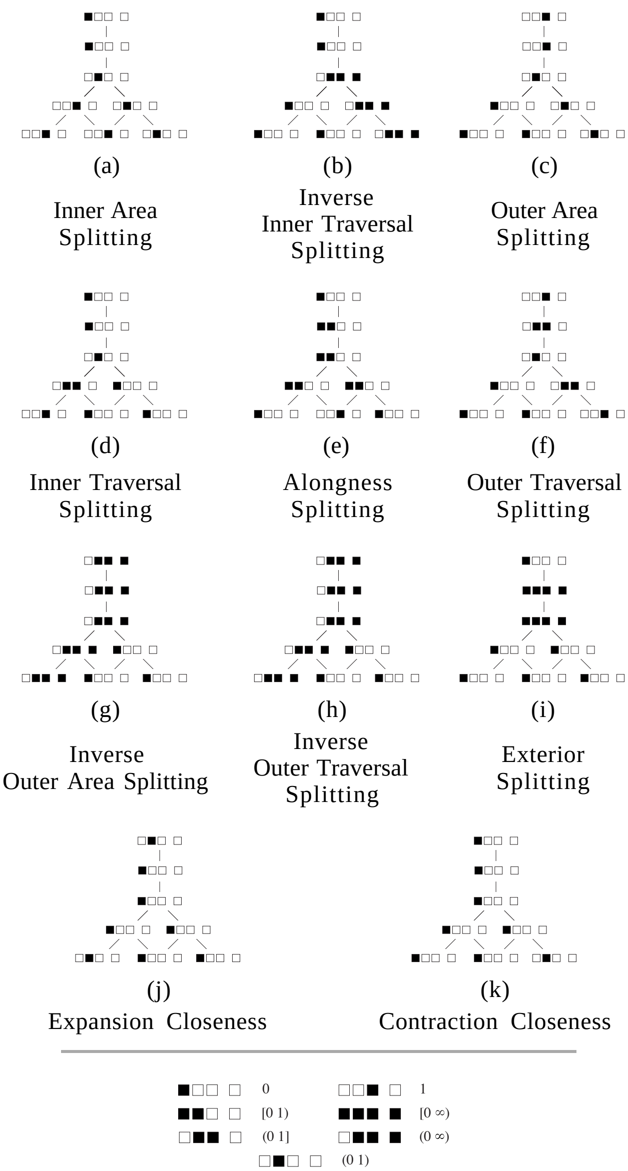


Figure 6. Value ranges of the eleven metric refinements for each of the eight topological relations between two regions, arranged by their conceptual neighborhood graph.

5. REFINEMENT SPECIFICATIONS TO TOPOLOGICAL RELATIONS

The goal is to derive uniquely each of the eight topological relations from one or more refinement specifications. A refinement specification that fully encapsulates the essence of a particular topological relation serves as the sole constraint powerful enough to imply that particular topological relation. This is, for instance, the case with AS=1 as it only holds for *equal*, while the remaining eight relations require AS<1. No other relation than *equal* can be inferred form AS=1. Such an inference from a single constraint only applies to one other refinement specification as 0<CC<1 implies *contains*, since for the remaining seven relations CC must be 0. Beside *equal* and *contains*, however, no other topological relation can be inferred from a single refinement specification.

The conjunction of two refinement specifications offers a further opportunity to make unique topological inferences. For instance, if both 0<IAS<1 and OTS=1 hold for the same configuration then that configuration must be the topological relations *overlap* or *covers*. This inference results from the intersection of the relations that respond to the constraints imposed by the two refinement specifications, that is, {*overlap*, *covers*, *contains*} for 0<IAS<1 and {*meet*, *overlap*, *covers*} for OTS=1. Of particular interest are those intersections that yield a single relation (e.g., *overlap* is the only relation that fulfills 0<IAS<1 *and* 0<ITS<1).

The following questions are addressed subsequently.

Q1: Can all eight topological relations be uniquely determined from refinement specifications?
Q2: Can all eight binary topological relations other than *equal* and *contains* be uniquely determined by a pair of refinement specifications, or does the unique inference of some relations require more than two refinement specifications?
Q3: Do all metric refinements contribute to uniquely determining topological relations?

5.1 Same Refinement Specification

In order to enable topological inferences from metric refinements, the reverse mappings—from refinement specifications onto applicable topological relations—are needed. These reverse mappings are first arranged by common refinement specifications, grouped for each topological relation. This approach establishes for each topological relation the set of relations that respond to the same refinement specification and is captured by the mapping τ.

For instance, *disjoint* has the value range IAS=0, which applies to one other candidate relation (*meet*), therefore, $\tau[\mathit{disjoint}, \text{IAS}=0] = \{\mathit{disjoint}, \mathit{meet}\}$. On the other hand, *overlap* shares the value range 0<ITS<1 with *coveredBy*, because *coveredBy*'s range 0<ITS<1 intersects with overlap's range 0<ITS≤1, therefore, $\tau[\mathit{overlap}, 0<\text{ITS}<1] = \{\mathit{overlap}, \mathit{coveredBy}\}$. Reversely, *coveredBy* offers two constraints from its value range 0<ITS≤1—first 0<ITS<1, which applies to *coveredBy* (0<ITS≤1), therefore, $\tau[\mathit{coveredBy}, 0<\text{ITS}<1] = \{\mathit{overlap}, \mathit{coveredBy}\}$, and second ITS=1, which applies to *coveredBy* (0<ITS≤1) as well as *inside* (ITS=1), therefore, $\tau[\mathit{coveredBy}, \text{ITS}=1] = \{\mathit{coveredBy}, \mathit{inside}\}$. A range that is not applicable for a particular topological relation would result in the empty candidate relation, for instance $\tau[\mathit{equal}, CC=1] = \varnothing$.

For the presentation of a relation's τ–tuples we choose an iconic display that highlights on the eight relations' conceptual neighborhood graph [6] all those relations that apply to a chosen refinement specification to support the visual confirmation for the intersection of a relation's refinement specifications. Of greatest interest are those intersections that result in a singleton, yielding a unique inference. For instance, the intersection of IAS=0 with 0<EC<1 yields a single relation, *disjoint* (Figure 7a). In two cases a single refinement specification is sufficient to determine uniquely a topological relation (Figure 7b).

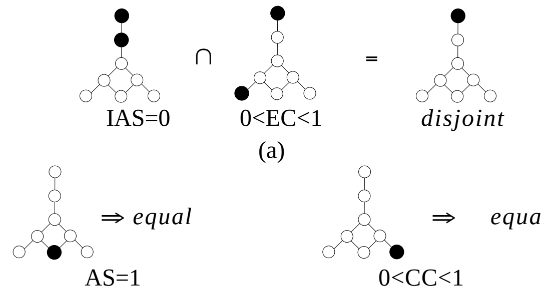
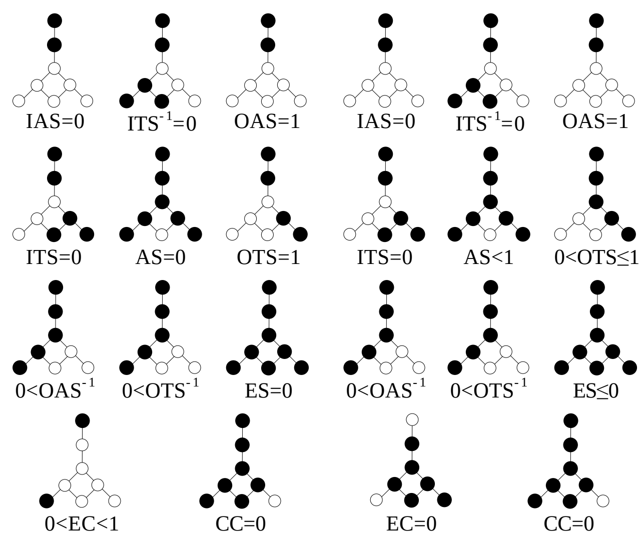
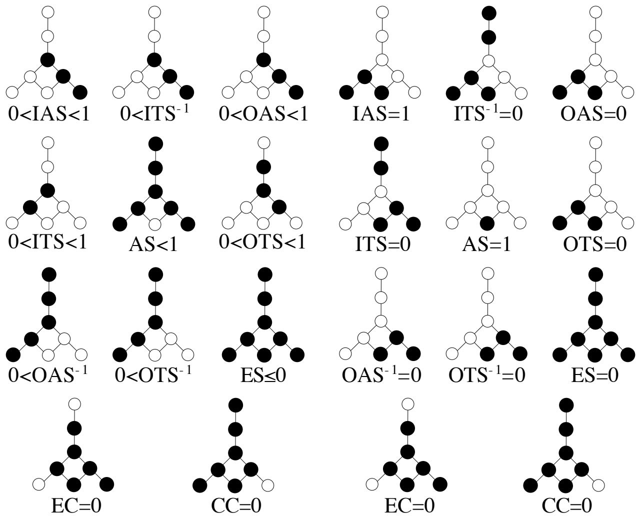


Figure 7. Visualization of applicable relations for refinement specifications: (a) intersection that yields a single topological relation and (b) single refinement specifications that imply a unique topological relation.

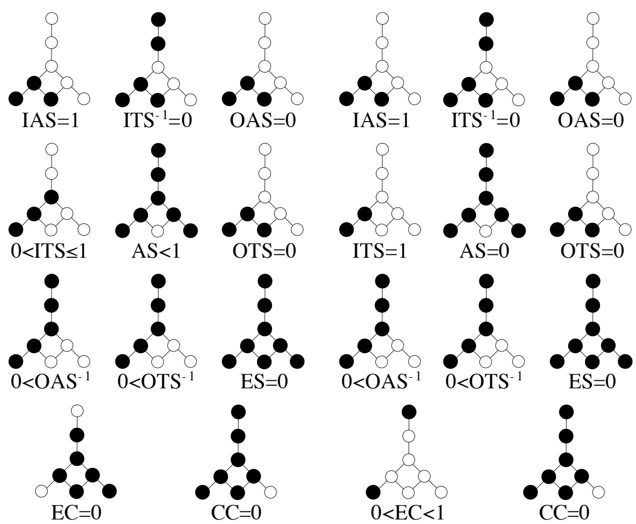
Figure 8 groups all τ–tuples that apply to a particular relation. An exhaustive combinatorial intersection of all refinement specifications determines what conjunctions of refinement specifications determine uniquely a topological relation (Section 5.3).



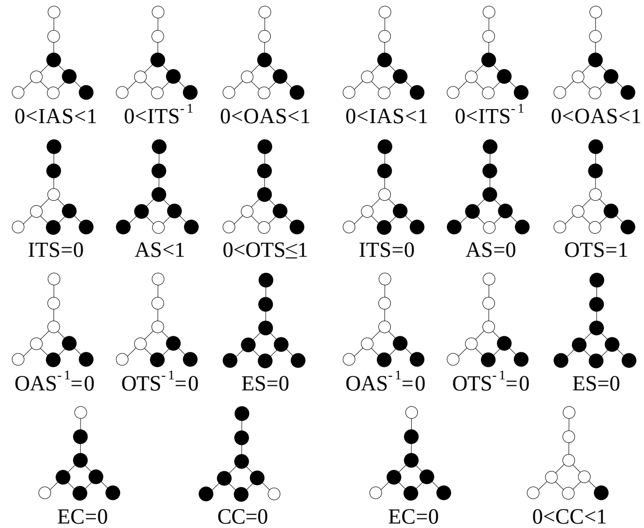
(a) $\tau[\mathit{disjoint},*]$ (b) $\tau[\mathit{meet},*]$



(c) τ[*overlap*,*] (d) τ[*equal*,*]



(e) τ[*coveredBy*,*] (f) τ[*inside*,*]



(g) τ[*covers*,*] (h) τ[*contains*,*]

Figure 8. τ–tuples that apply to a particular relation such that in each panel the splitting ratios are arranged spatially according to the underlying 9-intersection, while the two closeness measures are placed underneath.

5.2 Pairs of Refinement Specifications that Imply Unique Topological Relations

The intersections of τ–tuples reveal that 30 pairs and two single refinement specifications define uniquely a topological relation (Figure 9).

This finding provides answers to two of the three questions (Q1–Q3) posed at the beginning of Section 5.

Q1: Six topological relations are uniquely defined by single pairs of refinement specifications, but the two relations *coveredBy* and *covers* fail to respond to a unique determination from one or two refinements.

Q3: The 62 (2*30+2*1) refinement specifications build on ten of the eleven metric refinements—only Exterior Splitting (ES) does not contribute to a unique inference. The reason for ES's deficiency is that ES>0 as the discriminating characteristic applies to *meet* and *overlap*, but it does not for every *meet* or *overlap* scenario. For example, a simple *meet* with a single-component boundary-boundary intersection has ES=0, which does not distinguish it from the other relations (except if *overlap* had ES>0, but like for *meet*, this is not a necessary condition for *overlap*). The most frequently occurring metric refinements are EC (10), IAS (9), and OAS and ITS (both 8), with 0<EC<1 being the most frequent refinement specification (8 times).

$IAS=0 \wedge 0<EC<1 \Rightarrow$ *disjoint*	$0<ITS^{-1} \wedge 0<OAS^{-1} \Rightarrow$ *overlap*
$ITS^{-1}=0 \wedge OTS^{-1}=0 \Rightarrow$ *equal*	$OAS=1 \wedge 0<EC<1 \Rightarrow$ *disjoint*
$0<ITS^{-1} \wedge 0<OTS^{-1} \Rightarrow$ *overlap*	$OAS=0 \wedge OAS^{-1}=0 \Rightarrow$ *equal*
$ITS=0 \wedge 0<EC<1 \Rightarrow$ *disjoint*	$0<OAS<1 \wedge 0<ITS<1 \Rightarrow$ *overlap*
$OAS=0 \wedge OTS^{-1}=0 \Rightarrow$ *equal*	$OTS=1 \wedge 0<EC<1 \Rightarrow$ *disjoint*
$0<OAS<1 \wedge 0<OAS^{-1} \Rightarrow$ *overlap*	$ITS=0 \wedge OTS=0 \Rightarrow$ *equal*
$IAS=0 \wedge EC=0 \Rightarrow$ *meet*	$0<OAS<1 \wedge 0<OTS^{-1} \Rightarrow$ *overlap*
$OTS=0 \wedge OAS^{-1}=0 \Rightarrow$ *equal*	$OAS=1 \wedge EC=0 \Rightarrow$ *meet*
$0<ITS<1 \wedge 0<OTS<1 \Rightarrow$ *overlap*	$OTS=0 \wedge OTS^{-1}=0 \Rightarrow$ *equal*
$0<IAS<1 \wedge 0<ITS<1 \Rightarrow$ *overlap*	$IAS=1 \wedge ITS=0 \Rightarrow$ *equal*
$IAS=1 \wedge 0<EC<1 \Rightarrow$ *inside*	$0<IAS<1 \wedge 0<OAS^{-1} \Rightarrow$ *overlap*
$IAS=1 \wedge OAS^{-1}=0 \Rightarrow$ *equal*	$OAS=0 \wedge 0<EC<1 \Rightarrow$ *inside*
$0<IAS<1 \wedge 0<OTS^{-1} \Rightarrow$ *overlap*	$IAS=1 \wedge OTS^{-1}=0 \Rightarrow$ *equal*
$ITS=1 \wedge 0<EC<1 \Rightarrow$ *inside*	$0<ITS^{-1} \wedge 0<ITS<1 \Rightarrow$ *overlap*
$ITS^{-1}=0 \wedge OAS^{-1}=0 \Rightarrow$ *equal*	$OTS=0 \wedge 0<EC<1 \Rightarrow$ *inside*
$AS=1 \Rightarrow$ *equal*	$0<CC<1 \Rightarrow$ *contains*

Figure 9. Thirty pairs and two single refinement specifications that uniquely define a topological relation.

The failure of determining uniquely *coveredBy* and *covers* with a pair of refinement specifications begs the question whether these two relations can be derived at all from conjunctions of more than three refinement specifications. The intersection of all τ[*coveredBy*,*] (Figure 8e) yields *coveredBy*. Likewise, the intersection of all τ[*covers*,*] (Figure 8g) yields *covers*. Therefore, it is feasible to determine both relations with a conjunction of more than two refinement specifications.

For *coveredBy* EC=0 must belong to the constraining set as it is the only refinement specification that eliminates *inside*. Also one of $0<ITS\leq1$, $AS\leq0$, $0<OTS^{-1}$, and $0<OAS^{-1}$ must be included to prune *equal*. With IAS=1, OAS=0, or OTS=0 as the third condition *coveredBy* is uniquely inferred by a triple of refinement specifications. A similar approach finds the three specifications CC=0, $OAS^{-1}=0$ or $OTS^{-1}=0$, and any of $0<IAS<1$, O<ITS-1, $0<OAS<1$, $0\leq AS<1$, $0<OTS\leq1$ to determine uniquely *covers*. So *covers* and *coveredBy* add twenty-two 3-term conjunctions of refinement specifications to the thirty 2-term conjunctions and two single-term refinement specifications (Figure 9).

5.3 Eliminating Redundancies among Refinement Specifications

Two refinement specifications would be redundant if they apply to exactly the same set of relations. Such redundant specifications inflate the number of necessary refinement specifications and, therefore, can be eliminated without loosing expressive power. Among the 88 refinement specifications (Figure 8a–h) are seven biconditional pairs (Equations 3a-g) so that only one of these pairs contributes to a smallest set of constraints. By transitivity another two dependencies—between $0<ITS^{-1}$ and $0<OAS<1$ (Equations 3b and 3d) and between OTS=0 and OAS=0 (Equations 3c with 3e)—are established. The dependency of all IAS and OAS values (Equations 3a–3c) was already observed at the level of the refinement formalisms (Equation 1), while the other four dependencies are not so obvious from the refinement formalisms.

$$IAS = 0 \Leftrightarrow OAS = 1 \tag{3a}$$

$$0 < IAS < 1 \Leftrightarrow 0 < OAS < 1 \tag{3b}$$

$$IAS = 1 \Leftrightarrow OAS = 0 \tag{3c}$$

$$0 < IAS < 1 \Leftrightarrow 0 < ITS^{-1} \tag{3d}$$

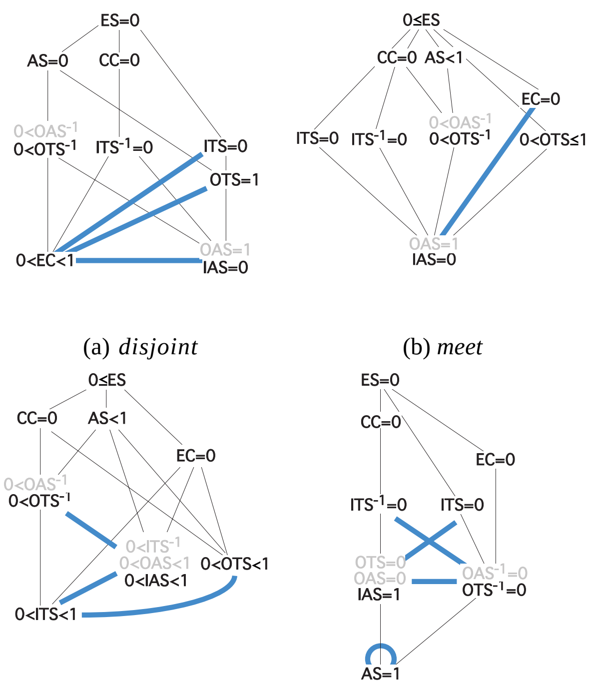
$$IAS = 1 \Leftrightarrow OTS = 0 \tag{3e}$$

$$OTS^{-1} = 0 \Leftrightarrow OAS^{-1} = 0 \tag{3f}$$

$$0 < OTS^{-1} \Leftrightarrow 0 < OAS^{-1} \tag{3g}$$

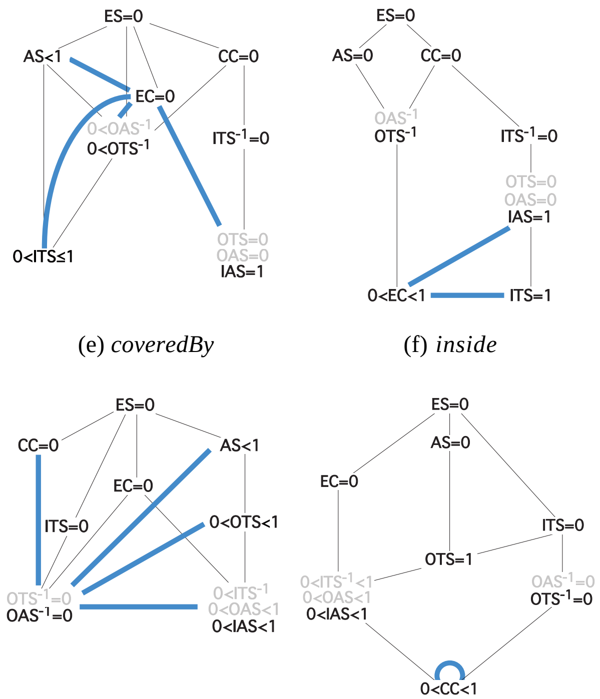
The iconic representation of the τ–tuples also supports the visual analysis of specification *subsumption*. A refinement specification subsumes another if the set of relations that respond to it are a superset of all relations that respond to the other refinement specification. For instance, {*equal*, *coveredBy*, *inside*} respond to IAS=1, while {*coveredBy*, *inside*} respond to ITS=1; therefore, IAS=1 subsumes ITS=1.

Subsumption graphs—essentially Hasse diagrams of the refinement specifications—depict for each topological relation the dependencies among refinement specifications (Figure 10). A pair of refinement specifications that infers a single topological relation implies that its subsumed specifications also yield the same unique inference, which sufficiently defines a relation. For instance, 0<EC<1 and ITS=0 yield *disjoint* (Figure 10a). Since ITS=0 subsumes OTS=1, OAS=1, and IAS=0, their conjunctions with 0<EC<1 must yield *disjoint* as well; therefore, ITS=0 *and* 0<EC<1 are the most encompassing specifications to infer *disjoint*. The superimposition of the 20 defining refinement specifications (Section 5.2) onto the subsumption graphs highlights subsumptions of these critical pairs and triples.



(a) *disjoint* (b) *meet*

(c) *overlap* (d) *equal*



(e) *coveredBy* (f) *inside*

(g) *covers* (h) *contains*

Figure 10. Subsumption graphs for each topological relation's the refinement specifications, with the least specific refinement at the top and the most specific refinements at the bottom. Redundant specifications are grayed out.

5.4 Fewest Refinement Specifications

With the subsumptions the set of 20 defining refinement specifications can be further pruned to a smallest set of refinement specifications. In some cases the membership in this set is obvious. For example, 0<CC<1 is required because it is the only defining specification of *contains*. Likewise, IAS=0 and EC=0 are both required, because they form the only pair of defining specifications for *meet*. Likewise CC=0 and OTS^{-1}=0 are included in all 3-way specifications for *covers*, therefore, they must be in the smallest set as well. Figure 11 compiles the eleven refinement specifications that are sufficient to completely define all eight topological relations. The eleven refinement specifications resort to six of the eleven metric refinements.

IAS=0	ITS=0	0<OTS^{-1}	CC=0
0<IAS<1	AS<1	EC=0	0<CC<1
IAS=1	OTS^{-1}=0	0<EC<1	

Figure 11. The smallest set of refinement specifications necessary to infer all eight topological relations.

6. CONCLUSIONS

The interaction between topology and metric were investigated with the goal of deriving uniquely from metric refinements topological relations. A set of eleven metric refinements, consisting of splitting measures and closeness measures, were defined. Splitting measures apply to non-empty intersections of interiors, boundaries, and exterior, while closeness measures apply to empty boundary-boundary intersections. Together with the possible value ranges that characterize each topological relation the metric refinements create refinement specifications. We derived that each topological relation can be determined from the conjunction of up to three such refinement specifications. The relations *equal* and *meet* can be derived from a single refinement specification, while *coveredBy* and *covers* require three. The remaining four relations are fully defined with two refinement specifications. We also derived the smallest set of these specifications that is sufficient to determine all relations.

These dependencies between metric and topological relations open the door for a number of future investigations. With the known compositions about topological relations as the benchmark it is now possible to determine the composition table of the metric refinements. This will enable metric reasoning without the need of direction information. The extension to the eleven topological relations on the sphere should provide further insights about how these metric refinements generalize. Finally the combined metric and topological units provide the opportunity for the definition of a new combined alphabet for spatial reasoning that starts with the topological *and* metric properties.

7. ACKNOWLEDGMENTS

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